The Thirtieth Annual
Eastern Shore High School Mathematics Competition

November 7, 2013

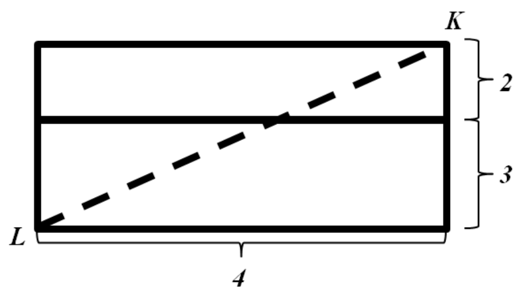
Team Contest Exam Solutions

1. You may use the given rectangular solid to help you solve this problem.

   The dimensions of a rectangular solid are 2, 3 and 4 units. Determine the length of the shortest path that meets the following two conditions:

   (a) The path connects a pair of opposite vertices.
   (b) The path can be drawn on the surface of the solid.

   Solution:



$$KL = \sqrt{(2+3)^2 + 4^2} = \sqrt{41} \text{ units}$$

2. Note that the set of **positive integers** is $\{1, 2, 3, 4, 5, \dots\}$.

Although $\sqrt{2} + \sqrt{3}$ does not equal the square root of a positive integer, $\sqrt{27} + \sqrt{48}$ does.

(a) Find a positive integer n such that $\sqrt{27} + \sqrt{48} = \sqrt{n}$.
You must provide written work to show why $\sqrt{27} + \sqrt{48} = \sqrt{n}$.

(b) Find another example like the one in part (a).
In other words, find positive integers a, b, and c such that $\sqrt{a} + \sqrt{b} = \sqrt{c}$. a, b, and c cannot be square numbers.
You must provide written work to justify your answer.

(c) Find every set of positive integers a, b, and c such that

- a, b, and c are all less than 35
- $\sqrt{a} + \sqrt{b} = \sqrt{c}$
- a, b, and c are distinct
- None of $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$ is an integer.

Solution:
(a) $\sqrt{27} + \sqrt{48}$

$= \sqrt{3^2 \cdot 3} + \sqrt{4^2 \cdot 3}$

$= 3\sqrt{3} + 4\sqrt{3}$

$= 7\sqrt{3}$

$= \sqrt{7^2 \cdot 3}$

$= \sqrt{147}$

(b) $\sqrt{50} = \sqrt{5^2 \cdot 2} = 5\sqrt{2} = \sqrt{2} + 4\sqrt{2} = \sqrt{2} + \sqrt{32}$

(c) From parts (a) and (b), notice that c must be of the form nk^2 where n and k are integers greater than 1. We can list all such numbers that are less than 35 as follows:

k	values of nk^2
2	8, 12, 16, 20, 24, 28, 32
3	18, 27
4	32

16 can be ruled out since $\sqrt{16}$ is an integer. 8 can also be ruled out. That is, since $\sqrt{8} = 2\sqrt{2} = \sqrt{2} + \sqrt{2}$, a would equal b which is not allowed. Similarly, since $\sqrt{12} = 2\sqrt{3}$, $\sqrt{20} = 2\sqrt{5}$, $\sqrt{24} = 2\sqrt{6}$, and $\sqrt{28} = 2\sqrt{7}$, 12, 20, 24, and 28 can be ruled out.

The only remaining possibilities for c are 18, 27, and 32. Each case is shown below.

$\sqrt{18} = 3\sqrt{2} = \sqrt{2} + 2\sqrt{2} = \sqrt{2} + \sqrt{8}$

$\sqrt{27} = 3\sqrt{3} = \sqrt{3} + 2\sqrt{3} = \sqrt{3} + \sqrt{12}$

$\sqrt{32} = 4\sqrt{2} = \sqrt{2} + 3\sqrt{2} = \sqrt{2} + \sqrt{18}$

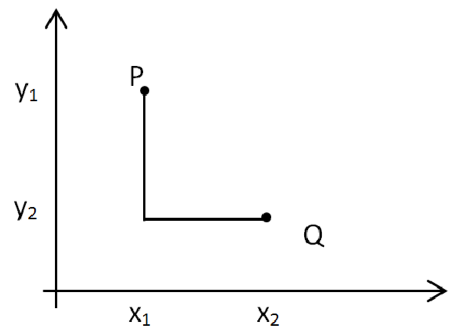
The total solution set is shown in the table.

a	b	c
2	8	18
3	12	27
2	18	32

3. For two points, $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in a coordinate plane, the **Taxicab Distance** between P and Q is

$$d(P, Q) = |x_2 - x_1| + |y_2 - y_1|$$

Illustrated below, it measures the distance as a taxicab would travel on a rectangular grid of city streets.



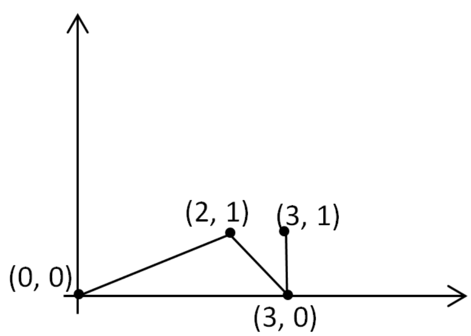
(a) Find $d(R, S)$ if $R = (1, 6)$ and $S = (4, 2)$.

A **Taxicab Loop** is the set of all points that have the same Taxicab Distance from a given point, referred to as the **Dispatch Point**. The Taxicab Distance from the Dispatch Point to any point on the Loop is known as the **Radius of the Taxicab Loop**.

(b) Sketch a Taxicab Loop with Dispatch Point at (0, 0) and with Radius 1.

The **Taxicab Length** of a path consisting of straight line segments is the total of the Taxicab Distances from one segment endpoint to the next on the path.

For instance, the Taxicab Length of the path illustrated below is 3 + 2 + 1 = 6.

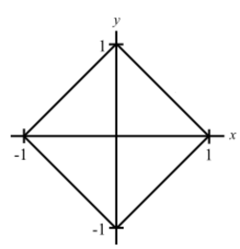


(c) Can we use this to calculate the length of a Taxicab Loop? If not, explain why not. Otherwise, calculate the ratio of the length of a Taxicab Loop to the Radius of the Loop.

Solution:

(a) $d(R, S) = |1 - 4| + |2 - 6| = 7$.

(b) A point (x, y) on a Taxicab Loop with Dispatch Point $(0, 0)$ and Radius 1 must satisfy the equation $|x-0|+|y-0| = 1$ or $|x| + |y| = 1$. A sketch of $|x| + |y| = 1$ is shown below.



(c) Consider a sketch of the Taxicab Loop as in part (b). The Taxicab Length of a Taxicab Loop with Radius r is

$$d((-r, 0), (0, r)) + d((0, r), (r, 0)) + d((r, 0), (0, -r)) + d((0, -r), (-r, 0)) = 2r + 2r + 2r + 2r = 8r$$

Therefore, the ratio of the length of a Taxicab Loop to the Radius of the Loop is 8.