The Twenty-Ninth Annual
Eastern Shore High School Mathematics Competition

November 8, 2012

Team Contest Exam Solutions

1. In a recent mathematics department meeting at WLM University, the 33 members present ranked who they thought had invented calculus. The members were all required to choose exactly one first place candidate, one second place candidate and one third place candidate. The three candidates to be voted upon were Leibniz, Newton, and the famous Jerk (otherwise known as the third derivative of position with respect to time).

(a) There are n ways that each voter can rank the candidates. Find n.

(a) Solution Since there are 3 candidates, and each candidate must be chosen exactly once for a ranking position we have 3 x 2 x 1= 3! = 6 possible rankings as shown below.

J, L, N
J, N, L
N, J, L
N, L, J
L, J, N
L, N, J

(b) Each member is given 3 points to distribute to the candidates (2 points for first choice and 1 point for second choice). Find the number of points Newton, Leibniz, and the Jerk each receive if the votes were distributed amongst three of the rankings as shown in the following table:

Rank	19 voters	10 voters	4 voters
1st	L	J	N
2nd	N	N	J
3rd	J	L	L

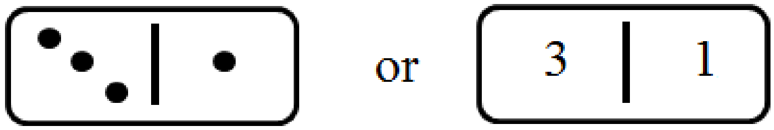
(b) Solution.
Newton would receive 19 x 1 pt + 10 x 1 pt + 4 x 2 pts = 37 pts
Leibniz would receive 19 x 2 pts + 14 x 0 pts = 38 pts
The Jerk would receive 10 x 2 pts + 4 x 1 pt = 24 pts

(c) Is there a way to distribute the 33 votes such that L wins the most first place rankings BUT does not earn the most points? If there is, create a table (similar to the one above) for such a situation and explain how the table justifies your answer.

(c) Solution. One possiblility is the following:

Rank	19 voters	8 voters	6 voters
1st	L	J	N
2nd	N	N	J
3rd	J	L	L

2. Consider a set of "double-N dominoes." Each domino has two regions separated by a bar. The dots which appear on dominoes are also called pips. Each region in a set of double-N dominoes contains one of the numbers in $\{0,1,\ldots,N\}$. A domino may be referred to by the numbers in the respective regions. The domino shown below is a 3-1 (or 1-3) domino.



Note: you have been given a set of double-3 dominoes.

- A double-0 set has 1 domino. i.e., the 0-0 domino.
- A double-1 set has 3 dominoes. i.e., the 0-0, 0-1 and 1-1 dominoes.
- A double-2 set has 6 dominoes. i.e., the 0-0, 0-1, 1-1, 0-2, 1-2 and 2-2 dominoes.

(a) How many dominoes are in a set of double-7 dominoes?

(b) Let N be a non-negative integer. How many dominoes are in a set of double-N dominoes? Express your answer in terms of N.

(a) and (b) Solution. There are 36 dominoes in a set of double-7 dominoes. There are $\frac{(N+1)(N+2)}{2}$ dominoes in a set of double-N dominoes. These answers can be found by listing the cases for $N = 1, 2, \ldots, 7$, as shown below, and then using the fact that the sum of the first K integers is $\frac{K(K+1)}{2}$ to generalize.

N = 0   0-0
D = 1   1 = 1

N = 1   0-0, 0-1, 1-1
D = 3   1 + 2 = 3

N = 2   0-0 , 0-1, 1-1, 0-2, 1-2, 2-2
D = 6   1 + 2 + 3 = 6

N = 3   0-0, 0-1, 1-1, 0-2, 1-2, 2-2, 0-3, 1-3, 2-3, 3-3
D = 10   1 + 2 + 3 + 4 = 10

N = 4   0-0, 0-1, 1-1, 0-2, 1-2, 2-2, 0-3, 1-3, 2-3, 3-3, 0-4, 1-4, 2-4, 3-4, 4-4
D = 15   1 + 2 + 3 + 4 + 5 = 15

N = 5   0-0 , 0-1, 1-1, 0-2, 1-2, 2-2, 0-3, 1-3, 2-3, 3-3, 0-4, 1-4, 2-4, 3-4, 4-4, 0-5, 1-5, 2-5, 3-5, 4-5, 5-5
D = 21   1 + 2 + 3 + 4 + 5 + 6 = 21

N = 6   0-0, 0-1, 1-1, 0-2, 1-2, 2-2, 0-3, 1-3, 2-3, 3-3, 0-4, 1-4, 2-4, 3-4, 4-4, 0-5, 1-5, 2-5, 3-5, 4-5, 5-5, 0-6, 1-6, 2-6, 3-6, 4-6, 5-6, 6-6
D = 28   1 + 2 + 3 + 4 + 5 + 6 + 7 = 28

N = 7   0-0, 0-1, 1-1, 0-2, 1-2, 2-2, 0-3, 1-3, 2-3, 3-3, 0-4, 1-4, 2-4, 3-4, 4-4, 0-5, 1-5, 2-5, 3-5, 4-5, 5-5, 0-6, 1-6, 2-6, 3-6, 4-6, 5-6, 6-6, 2-6, 3-6, 4-6, 5-6, 6-6, 0-7, 1-7, 2-7, 3-7, 4-7, 5-7, 6-7, 7-7
D = 36   1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36

Another way to examine double-N dominoes is to examine the sum of the pips. For example:

- In a double-0 set the sum of the pips is 0. i.e.,(0+0)=0.
- In a double-1 set the sum of the pips is 3. i.e.,(0+0)+(0+1)+(1+1)=3.
- In a double-2 set the sum of the pips is 12. i.e.,(0+0)+(0+1)+(1+1)+(0+2)+(1+2)+(2+2)=12.

(c) What is the sum of the pips in a set of double-7 dominoes?

(d) Let N be a non-negative integer. What is the sum of the pips in a set of double-N dominoes? Express your answer in terms of N.

(c) and (d) Solution. The sum of the pips in a set of double-7 dominoes is 252. The sum of the pips in a set of double-N dominoes is $N\left(\frac{(N+1)(N+2)}{2}\right)$. These answers can be found by listing the cases for $N = 1, 2, \ldots, 7$, as shown above. One can then make the following table:

N	number of dominoes	sum of the pips
0	1	0
1	3	3
2	6	12
3	10	30
4	15	60
5	21	105
6	28	168
7	36	252
$\ldots$	$\ldots$	$\ldots$
N	$\frac{(N+1)(N+2)}{2}$	$N\left(\frac{(N+1)(N+2)}{2}\right)$

3. (a) Find a 3-digit number with the following properties:

1. Each of the digits 1, 2, and 3 is used exactly once in the number.
2. If we ignore all but the first n digits of the number, we obtain a number divisible by n.

In other words, find a number abc so that the number abc is divisible by 3, the number ab is divisible by 2, and the number a is divisible by 1.

If no such number exists, give an explanation for why this is the case.

(a) Solution. One can easily check that 123 and 321 satisfy properties 1 and 2.

(b) Find a 4-digit number with the following properties:

1. Each of the digits 1, 2, 3, and 4 is used exactly once in the number.
2. If we ignore all but the first n digits of the number, we obtain a number divisible by n.

In other words, find a number $abcd$ so that the number $abcd$ is divisible by 4, the number abc is divisible by 3, the number ab is divisible by 2, and the number a is divisible by 1.

If no such number exists, give an explanation for why this is the case.

(b) Solution. The number must be divisible by 4, so the last digit must be even. The second digit must also be even so that property 2 is satisfied. Hence the only possibilites are 1234, 3214, 1432, and 3412.

1234 is not divisible by 4. To check divisibility by 4, one only need check the last two digits of a number. Since 4 does not divide 34, 4 does not divide 1234, so property 2 is not satisfied. Similarly, 4 does not divide 3214, so property 2 is not satisfied.

The only possiblilities left are 1432 and 3412. However, 3 does not divide 143 as $1 + 4 + 3 = 8$ is not divisible by 3, so property 2 is not satisfied. Similarly, 3 does not divide 341, so 3412 does not satisfy property 2.

Thus, since all possiblilities have been checked, there is no 4-digit number that satisfies both properties.

(c) Find a 5-digit number with the following properties:

1. Each of the digits 1, 2, 3, and 4, and 5 is used exactly once in the number.
2. If we ignore all but the first n digits of the number, we obtain a number divisible by n.

If no such number exists, give an explanation for why this is the case.

(c) Solution Any 5-digit number that satisfies both properties in part (c) must end in 5. Thus, the first four digits of the number must comprise a 4-digit number that satisfies properties 1 and 2 from part (b). From part (b), however, we know that no such 4-digit number exists. Thus, we cannot construct a 5-digit number that satisfies both properties in part (c).

(d) Find a 6-digit number with the following properties:

1. Each of the digits 1, 2, 3, and 4, 5, and 6 is used exactly once in the number.
2. If we ignore all but the first n digits of the number, we obtain a number divisible by n.

If no such number exists give an explanation for why this is the case.

(d) Solution As in part (b), we know that the second, fourth, and sixth digits must be even and, hence, the other digits must be odd. Also, the fifth digit must be 5 to satisfy the divisibility-by-5 requirement.

To satisfy the divisibility-by-4 requirement, the third and fourth digits must together comprise a number divisible by 4, as discussed in part (b). Thus, the only possibilites for digits 3, 4, and 5 are 125, 165, 325, and 365. Since the first digit must be odd, we can represent each of these situations as follows:

3_125_
3_165_
1_325_
1_365_

Notice that we cannot choose 4 for the second digit in 3_125_ because the resulting number would not satisfy the divisibility-by-3 requirement. Similarly, we cannot choose 6 for the second digit in 3_125_ . Hence, the case of 3_125_ cannot be completed to result in a 6-digit number that satisfies both properties. Similar reasoning eliminates the case of 1_325_.

A similar argument shows the only possible 6-digit numbers, constructed from the 3_165_ and 1_365_ cases, are 321654 and 123654. By construction, the divisibility-by-2, divisibility-by-4, and divisibility-by-5 requirements are satisified.

$1 + 2 + 3 = 6$, which is divisible by 3, so the divisibility-by-3 requirement is satisfied.

Also, $1 + 2 + 3 + 4 + 5 + 6 = 21$, which is divisible by 3. Thus, both 321654 and 123654 are divisible by 6, as each is divisible by both 2 and 3. Hence, 321654 and 123654 satisfy properties 1 and 2.