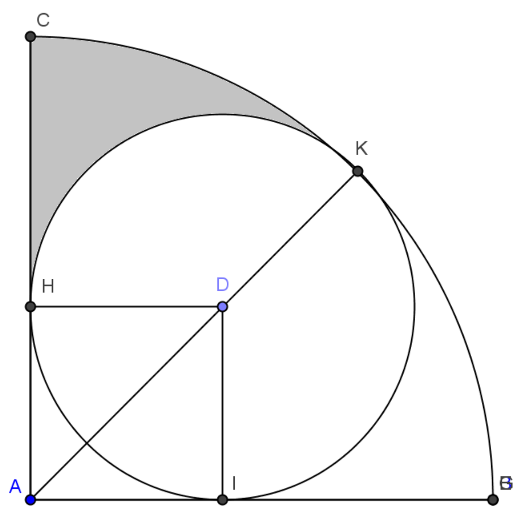
2011 Eastern Shore HSMC - Team Solutions

1. (a) Find the radius of the inscribed circle; (b) Find the area of the shaded region.

   (a) The radius of the inscribed circle is $\sqrt{2}-1$. To see this, construct a line through A and the center of the inscribed circle D and extend to the point of tangency of the two circles K. Note that DH, DI and DK are all radii of the inscribed circle and rectangle $AIDH$ is a square. If we call the radius of the inscribed circle r then the radius of the large circle is $r + r\sqrt{2}$, so $r + r\sqrt{2} = 1$ giving $r = \sqrt{2} - 1$.



   (b) For the area of the shaded region, we will let A_T represent the total area of the quarter circle, A_1 the area of the shaded region, A_2 the area of the enclosed region close to the point A and A_C the area of the inscribed circle. Then $A_T = 2A_1 + A_2 + A_C$, giving $A_1 = \frac{1}{2}(A_T - A_2 - A_C)$. Since the radius of the inscribed circle is $r = \sqrt{2} - 1$, we have

$$\begin{aligned}
A_C &= \pi\left(\sqrt{2}-1\right)^2 \\
A_2 &= \left(\sqrt{2}-1\right)^2 - \frac{1}{4}\pi\left(\sqrt{2}-1\right)^2 \\
&= \left(\sqrt{2}-1\right)^2\left(1-\frac{\pi}{4}\right) \\
A_T &= \frac{\pi}{4}
\end{aligned}$$

So,

$$\begin{aligned} A_1 &= \frac{1}{2}\left(\frac{\pi}{4} - \left(\sqrt{2}-1\right)^2\left(1-\frac{\pi}{4}\right) - \pi\left(\sqrt{2}-1\right)^2\right) \\ &= -\frac{3}{2} + \sqrt{2} + \frac{\pi}{4}(-4+3\sqrt{2}) \\ &= -\frac{3}{2} + \sqrt{2} - \pi + \frac{3\pi}{2\sqrt{2}} \end{aligned}$$

2. (a) Find the least integer greater than 3 which leaves a remainder of 3 when divided by each of the integers 4 through 10, inclusive.

   (b) Find the least integer greater than 3 which is divisible by 11 *and* leaves a remainder of 3 when divided by each of the integers 4 through 10, inclusive.

   Solutions.

   (a) For a number to leave a multiple of 3 when divided by each of the integers 4 through 10, it must be three more than a common multiple of the integers 4 through 10. To find the least such integer, we must find the least common multiple of the integers 4 through 10. The least common multiple can be determined from the prime factorizations of these integers: it must be divisible by 2^3 (since $8 = 2^3$), 3^2 (since $9 = 3^2$), 5, and 7. So, $LCM(4,5,6,7,8,9,10) = 2^3 \cdot 3^2 \times 5 \times 7 = 2520$. Thus, the least integer greater than 3 that leaves a remainder of 3 when divided by each of the integers 4 through 10, inclusive, is 2523.

   (b) From part (a), we know that a positive integer will leave a remainder of 3 when divided by each of the integers 4 through 10 if (and only if) it is 3 more than a multiple of 2520. So, we are looking for a number of the form $2520n + 3$ such that n is a positive integer and $2520n + 3$ is a multiple of 11.

   The least such integer can be found through trial-and-error. For a more direct method of solution, note that 2520 itself is 1 more than a multiple of 11 (since $2519 = 11 \times 229$). It follows that 8×2520 will be 8 more than a multiple of 11; so, $8 \times 2520 + 3$ will be 8+3=11 more than a multiple of 11, thus a multiple of 11 itself. Therefore, our answer is $8 \times 2520 + 3 = 20163$.

3. Two cars (we'll call them car A and car B) are driving towards each other at a constant speed of 60 miles per hour. At the instant that the cars are 120 miles apart, a bird which had been sitting on car A flies on ahead toward car B; the bird flies at a constant speed of 70 miles per hour. When the bird reaches car B, it immediately turns around and flies back toward car A; thereafter the bird continues to fly back and forth between the two cars, always maintaining its speed of 70 miles per hour.

   Find the total distance traveled by the bird, from the instant it initially leaves car A until the instant cars A and B pass each other.

Solution: The cars will pass each other in exactly one hour. Since the bird's speed is a constant 70 miles per hour, it will travel a total of 70 miles.

..or...

Starting from car A, the bird's speed relative to car B is 70-(-60)=130 mph. Since they start 120 miles apart, the bird will reach car B in $\frac{120}{130} = \frac{12}{13}$ of an hour, during which time it will travel $70 \times \frac{12}{13} = \frac{840}{13}$ miles. At the time the bird reaches car B, the cars have each traveled $60 \times \frac{12}{13} = \frac{720}{13}$ miles, meaning the distance between them is now $120 - 2 \times \frac{720}{13} = \frac{1560}{13} - \frac{1440}{13} = \frac{120}{13}$ miles.

Now, starting from car B, the bird flies back toward car A. Its speed relative to car A is 130 mph. Since they start 120/13 miles apart, the bird will reach car B in $\frac{(120/13)}{130} = \frac{12}{13^2}$ of an hour, during which time it will travel $70 \times \frac{12}{13^2} = \frac{840}{13^2}$ miles. At the time the bird reaches car B, the cars have each traveled $60 \times \frac{12}{13^2} = \frac{720}{13^2}$ miles, meaning the distance between them is now $\frac{120}{13} - 2 \times \frac{720}{13^2} = \frac{1560}{13^2} - \frac{1440}{13^2} = \frac{120}{13^2}$ miles.

At this point we can observe a developing pattern. Each time the bird flies from one car to the other, the distance between the cars (and the corresponding distance flown by the bird) is reduced to 1/13 of the previous distance. This implies the bird will fly first $\frac{840}{13}$ miles, then $\frac{840}{13^2}$ miles, then $\frac{840}{13^3}$ miles, and so on. To find the total distance flown, then, we must evaluate the following seqence:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{840}{13^n} &= \frac{840}{13} + \frac{840}{13^2} + \frac{840}{13^3} + \dots \\
&= 840 \times \left(\frac{1}{13} + \frac{1}{13^2} + \frac{1}{13^3} + \dots \right) \\
&= 840 \times \frac{\frac{1}{13}}{1 - \frac{1}{13}} \\
&= 840 \times \frac{1/13}{12/13} \\
&= 840 \times \frac{1}{12} \\
&= 70
\end{aligned}$$

Thus, the bird flies a total of 70 miles.

Comment: this is an old, moderately well-known "trick question" among mathematicians, in the sense that there are two methods of solution, and mathematicians often fail to see the "easy" method until it's pointed out to them. A famous, amusing instance of this occurred when the brilliant early 20th century mathematician John von Neumann was posed with a similar problem. He solved the problem in his head almost instantly. (The following is paraphrased.) The person who posed the problem observed "That was very clever on your part; most people try to solve this problem by setting up an infinite series for the solution." Von Neumann, puzzled, replied: "What other way is there?"

4. (See the test for the definitions of "boxiness" and "boxulous.")

a) Find the 'boxiness' of every even positive integer less than or equal to 32:

Solution:

n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
box(n)	1	2	2	2	2	3	2	3	3	3	2	4	2	3	4	3

b) Based on your results and general observations while completing part (a), describe a way to find (without diagrams or tiles) positive even integers, n, such that box(n)=2.

Solution: If n is an even positive integer, then box(n)=2 if n is two times a prime number -or- if $n = 8$.

If $n = 2p$ for a prime p, the complete set of factors of n is $\{1, 2, p, 2p\}$, so the only "boxes" with area n have dimensions 1-by-$2p$ and 2-by-p.

Comment: The solution above is a complete solution when considering only *even* positive integers n. If we extended the same question to the complete set of positive integers (including odd integers), then there would be three types of number, n, such that box(n) = 2: $n = p^2$, $n = p^3$, and $n = pq$ (where p and q denote any two distinct primes). If $n = p^2$, its boxes are 1-by-p^2 and p-by-p. If $n = p^3$, its boxes are 1-by-p^3 and p-by-p^2. If $n = pq$, its boxes are 1-by-pq and p-by-q.

c) Evaluate each of the following: box(100), box(360), and box(2012).

Answers:

- Since $100 = 1 \times 100 = 2 \times 50 = 4 \times 25 = 5 \times 20 = 10 \times 10$, there are five boxes which could be formed with 100 tiles. Thus, box(100)=5.
- $360 = 1 \times 360 = 2 \times 180 = 3 \times 120 = 4 \times 90 = 5 \times 72 = 6 \times 60 = 8 \times 45 = 9 \times 40 = 10 \times 36 = 12 \times 30 = 15 \times 24 = 18 \times 20$. This means we could make twelve different boxes with 360 tiles. Thus, box(360)=12.
- The number 2012 is equal to 4×503. Since 503 is prime, it follows that there are only three boxes with 2012 tiles: 1-by-2012, 2-by-1006, and 4-by-503. Thus, box(2012)=3.

For parts (d) and (e), refer to the definition of "boxulous" on the test sheet.

d) Which even positive integer(s) less than or equal to 32, other than 6, is/are "boxulous?"

Answer: Among these numbers, only 28 is boxulous: $28 = 1 \times 28 = 2 \times 14 = 4 \times 7$, and $1 + 28 + 2 + 14 + 4 + 7 = 56$, which is two times 28.

e) Find a "boxulous" number that is greater than 32 but less than 1000. (Hint: there are three boxulous numbers between 32 and 1,000. All of them are even.)

Answers: The boxulous numbers between 32 and 1000 are 496, 120, and 672.

496: The sum of the dimensions of boxes with 496 tiles is 892; 892/496=2.

120: The sum of the dimensions of boxes with 120 tiles is 360; 360/120=3.

672: The sum of the dimensions of boxes with 672 tiles is 2016; 2016/672=3.

Comments: What you're really being asked to do in parts (d) and (e) - if we put it in numeric rather than geometric terms - is to find the sum of the factors of a positive integer, n, and determine whether that sum is a multiple of n itself.

The boxulous numbers 28 and 496 are examples of "perfect numbers." A perfect number is a positive integer, n, such that the sum of the factors of n (including n itself) is exactly $2n$. These are rare. (The next "perfect number" after 496 is 8128.)

The boxulous numbers 120 and 672 are called "3-multiperfect" numbers, because the sum of the factors of each number is three times the number itself. For example, the factors of 120 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120, which add up to exactly 360, which is 3×120.