The Twenty-Eighth Annual

Eastern Shore High School Mathematics Competition

November 10, 2011

Team Contest Exam

Instructions
Answer as many questions as possible in the time provided. To receive full credit for a correct solution, show all work and provide a clearly written explanation. Solutions will be judged based on correctness, completeness and clarity. (Little credit, if any, will be given for a solution consisting of just a number or a single sentence.)

All work and answers must be written on the provided sheets of plain white paper. Use only one side of each sheet of paper, and start each new problem on a new sheet of paper. Write your team name (that is, the name of the school which you are representing) at the top of each sheet that you turn in for scoring.

1. A circle is inscribed inside a quarter circle as pictured below. Assume that $\overline{AB}$ has a length of 1 unit. For each of the following problems, give an *exact* answer, rather than a decimal approximation.

    (a) Find the radius of the inscribed circle.
    (b) Find the area of the shaded region.
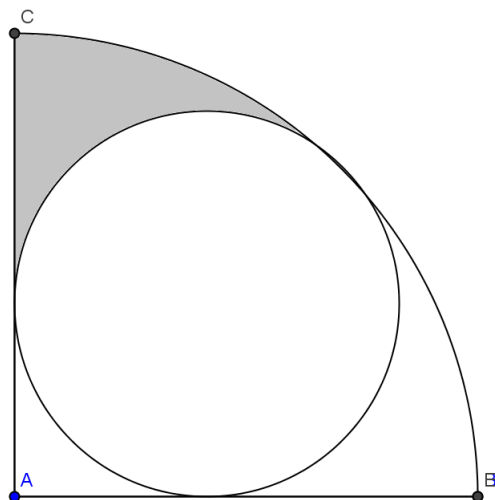
2. (a) Find the least integer greater than 3 which leaves a remainder of 3 when divided by each of the integers 4 through 10, inclusive.

    (b) Find the least integer greater than 3 which is divisible by 11 *and* leaves a remainder of 3 when divided by each of the integers 4 through 10, inclusive.

3. Two cars (we'll call them car A and car B) are driving towards each other at a constant speed of 60 miles per hour. At the instant that the cars are 120 miles apart, a bird which had been sitting on car A flies on ahead toward car B; the bird flies at a constant speed of 70 miles per hour. When the bird reaches car B, it immediately turns around and flies back toward car A; thereafter the bird continues to fly back and forth between the two cars, always maintaining its speed of 70 miles per hour.

   Find the total distance traveled by the bird, from the instant it initially leaves car A until the instant cars A and B pass each other.

4. For this problem, we define a new adjective, "boxiness," as follows: Given a positive integer n, the "boxiness" of n (abbreviated box(n)) is defined as the number of distinct rectangles, each consisting of unit squares, whose area is n. Here are a few examples to illustrate the "boxiness" concept:
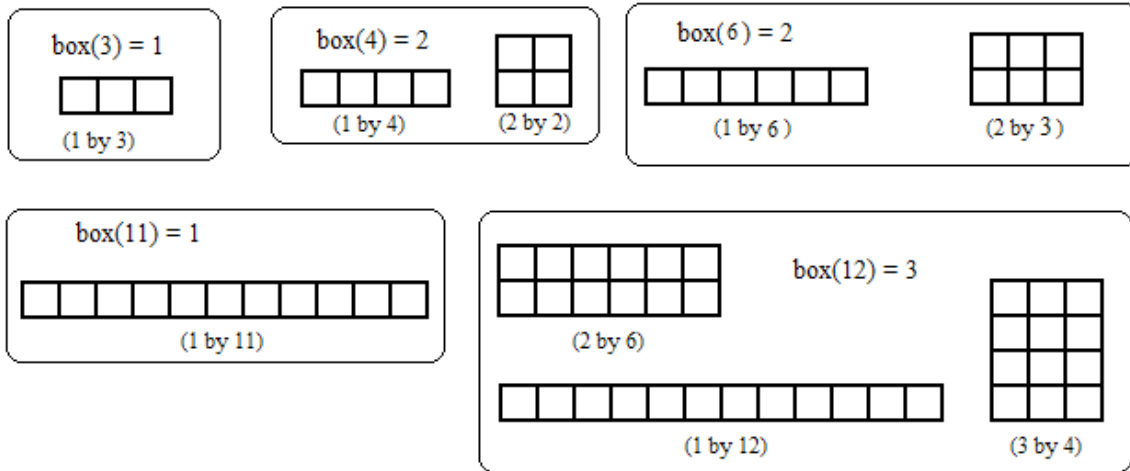
Solve each of the following problems; make use of the provided square tiles where appropriate. (In particular, you are encouraged to use the square tiles to help you to solve parts (a) and (d).)

a) Find the 'boxiness' of every even positive integer less than or equal to 32:

n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
box(n)	1	2	2			3										

b) Based on your results and general observations while completing part (a), describe a way to find (without diagrams or tiles) positive even integers, n, such that box(n)=2.

c) Evaluate each of the following: box(100), box(360), and box(2012).

For parts (d) and (e), refer to the following definition:

Definition: A positive integer, n, is called 'boxulous' if, when we list all of the distinct rectangles whose areas are n, the sum of the *dimensions* of all of the rectangles is a multiple of n. For example: 6 is "boxulous," because its rectangles' dimensions are 1-by-6 and 2-by-3; the sum of these dimensions is $1+6+2+3 = 12$, which is a multiple of 6. On ther other hand, 4 is not "boxulous," because the rectangles whose areas are 4 are 1-by-4 and 2-by-2; the sum of these dimensions is $1+4+2+2 = 9$, which is not a multiple of 4.

d) Which even positive integer(s) less than or equal to 32, other than 6, is/are "boxulous?"

e) Find a "boxulous" number that is greater than 32. (Hint: there are five boxulous numbers which are less than 1,000. All of them are even.)