The Twentieth Annual
Eastern Shore High School Mathematics Competition
Team Contest Solutions

1. Suppose player 1 starts by subtracting 9 from 29. 20 remains. The only squares that can be subtracted at this stage are 1, 4, 9, and 16.

If player 2 subtracts 16, then player 1 can subtract 4 to obtain 0.
If player 2 subtracts 4, then player 1 can subtract 16 to obain 0.
If player 2 subtracts 9, then player 1 can subtract 9 also, forcing player 2 to subtract 1. Player 1 can then win by subtracting 1.
If player 2 subtracts 1, there are several possibilities, but player 1 can guarantee success as shown below.

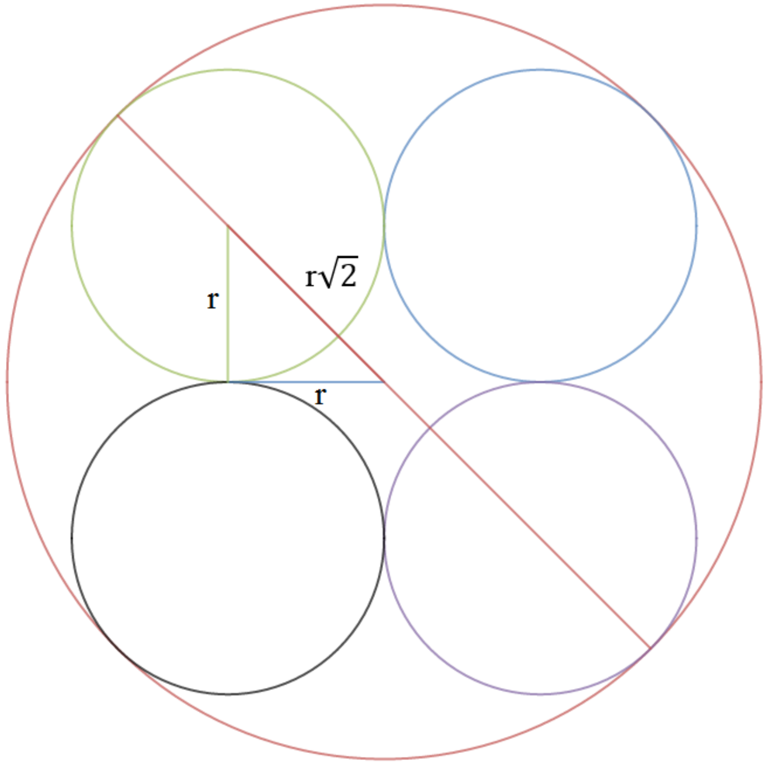
Player 1: 9
Player 2: 1
Player 1: 9
Player 2: 9
Player 1: 1

Player 1: 9
Player 2: 1
Player 1: 9
Player 2: 4
Player 1: 4
Player 2: 1
Player 1: 1

Player 1: 9
Player 2: 1
Player 1: 9
Player 2: 1
Player 1: 9

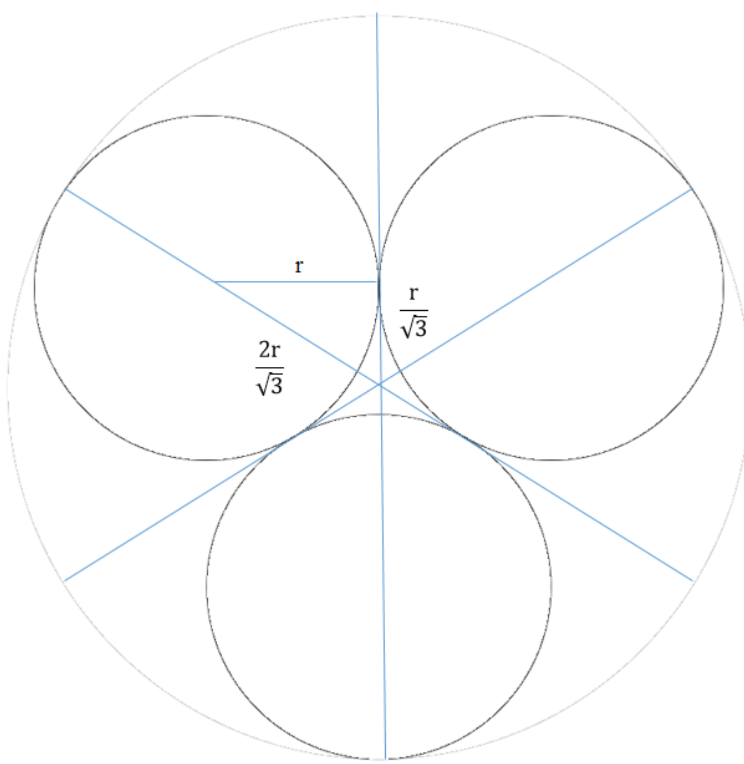
2. The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$, and the surface area is $4\pi r^2$. For the volume to be an integer, r must be a multiple of 3. When $r = 18$, the volume of the sphere is 7776π and the surface area is 1296π. This is the only solution, as it is easily verified that 15 is too small and 21 is too large.

3. a. The image below shows the optimal arrangement of the four circles within the given larger circle.



From this diagram, we see that the diameter of the large circle can be written as $4r + 2(r\sqrt{2} - r)$ where r is the radius of each smaller circle. This diameter is also 6 since the area of the large circle is 9π. So, $4r + 2(r\sqrt{2} - r) = 6$ which implies that $r = \dfrac{3}{1+\sqrt{2}} = 3(\sqrt{2} - 1)$.

b. The image below shows the optimal arrangement of the three circles within the given larger circle.



Using the fact that the triangle shown in the diagram is a 30-60-90 triangle, we can write the diameter as $2r+2\left(\frac{2}{\sqrt{3}}\right)r$. Again, this expression equals 6, so solving for r, we obtain that $r = \dfrac{3\sqrt{3}}{\sqrt{3}+2} = 3(2\sqrt{3}-3)$.

c. Using the answers from parts a and b, we can verify that there is less waste in the four circles case.

4. a. Edge AB coincides with the line $y = x + 1$. The altitude of edge AB passes through the point C and is perpendicular to AB. Any line perpendicular to $y = x+1$ must have slope -1, so we obtain that the equation of the altitude is $y = -x + 4$.

The midpoint of AB is given by $\left(\frac{0+3}{2}, \frac{1+4}{2}\right) = \left(\frac{3}{2}, \frac{5}{2}\right)$. The median of AB passes through this point and through C, so we can show that the equation of the median is also $y = -x + 4$.

The previous two results imply that $y = -x + 4$ must also be the equation of the perpendicular bisector of AB.

We could also note that AC and BC both have length $\sqrt{17}$, so triangle ABC is isosceles. This means that the altitude, median, and perpendicular bisector of AB are the same (since AC and BC are congruent), so once we have found the equation of the altitude, we are done.

b. The equation of the altitude is $y = -x + n$. The median passes through $\left(\frac{3}{2}, \frac{5}{2}\right)$ and $(n, 0)$, so we can show that the equation of the median is $y = \dfrac{5}{3-2n}(x-n)$. The equation of the perpendicular bisector is again $y = -x + 4$.

5. a.

R_1: $A \cap B' \cap C'$. Students taking only Mathematics.
R_2: $A \cap B \cap C'$. Students taking Mathematics and English but not Philosophy.
R_3: $A \cap B' \cap C$. Students taking Mathematics and Philosophy but not English.
R_4: $A \cap B \cap C$. Students taking all three courses.
R_5: $A' \cap B \cap C'$. Students taking only English.
R_6: $A' \cap B \cap C$. Students taking English and Philosophy but not Mathematics.
R_7 $A' \cap B' \cap C$. Students taking only Philosophy.
R_8: $A' \cap B' \cap C'$. Students taking none of the three courses.
R_9: $A' \cap W$. First semester female students who are not taking Mathematics.
R_{10}: $A \cap W$. First semester female students who are taking Mathematics.
R_{11}: $A \cap M$. First semester male students who are taking Mathematics.
R_{12}: $A' \cap M$. First semester male students who are not taking Mathematics.

b.

R_1: 10
R_2: 500
R_3: 100
R_4: 90
R_5: 100

R_6: 100
R_7: 100
R_8: 0
R_9: 100
R_{10}: 500
R_{11}: 200
R_{12}: 200