Eastern Shore High School Mathematics Competition

November 12, 2003

Team Contest

Answer as many questions as possible in the time provided. To receive full credit for a correct solution, show all work *and* provide a clearly written explanation. Solutions will be judged based on correctness, completeness and clarity. (Little credit, if any, will be given for a solution consisting of just a number or a single sentence.)

1. In the game "subtract-a-square", a positive integer is written down, and then two players alternately subtract integer squares from it until one player is able to leave zero, in which case that player is declared the winner. If the original number is 29, show that the first player has a winning strategy – that is, a strategy that will guarantee a win regardless of the choices made by the second player.

2. The surface area and volume of a sphere are both four-digit integer multiples of π. Find the radius of the sphere.

3. A machinist wishes to cut four equal circles from a circular piece of sheet metal whose area is 9π square feet. She wants the circles of metal to be the largest that can possibly be cut from the given piece.

   (a) Find the length of the radius of each of the four new circles.

   (b) If instead she were to cut three circles, find the length of the radius of each of the three new circles.

   (c) Which cut, three circles or four circles, leaves the least waste?

4. Find the equations of the altitude, median and perpendicular bisector of edge AB in the following triangles ABC. (Each triangle is defined in terms of its vertices in the xy-plane.)

   (a) The triangle with vertices A(0,1), B(3,4) and C(4,0).

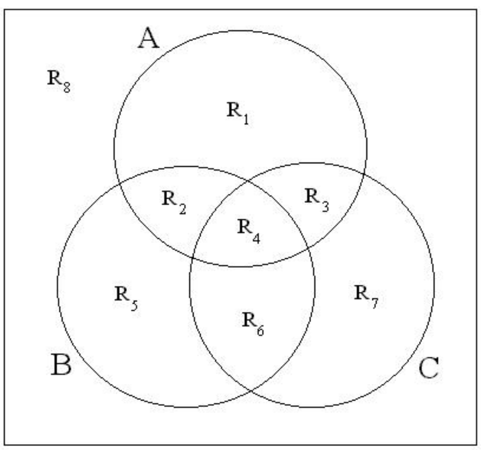
   (b) The triangle with vertices A(0,1), B(3,4) and C(n,0), where n is an arbitrary positive number.

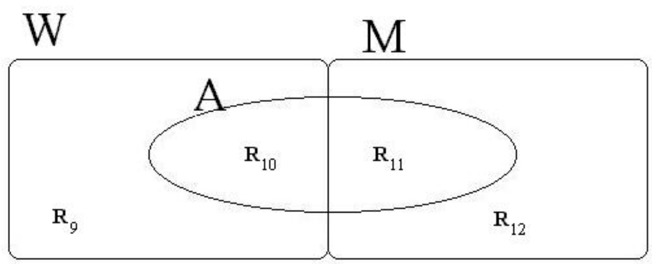
5. At Alachua Balaclava College (a.k.a. ABC), all first semester students are required to take at least one of the following courses: MATH 210 (Discrete Mathematics), ENGL 101 (Composition and Argument), or PHIL 100 (Basic Logic). No one but first semester students may enroll in these classes.

   Define the set A to be the set of students taking MATH 210, B the set of students taking ENGL 101, and C the set of students taking PHIL 100. Further, let W be the set of first semester female students and M the set of first semester male students.

   Below are Venn diagrams for A, B and C, and for A, W and M.

   (a) Describe each of the twelve regions (R_1, R_2, etc.) in the below Venn diagrams using natural language (e.g., "students taking Mathematics and English but not Philosophy") *and* using set notation (e.g., $A \cap B \cap C'$).





   (b) Suppose you know that there are 1000 first semester students at ABC, 600 of whom are women, 700 of whom are taking Discrete Mathematics, and 200 of whom are male Discrete Mathematics students. Further, suppose 90 of these students are taking all three first semester courses, 390 are taking PHIL 100, 790 are taking ENGL 101, and 590 are taking both Math and English. Finally, suppose there are 200 first semester students who either are taking only English or are taking all the first semester courses except English. Fill in each of the regions of both Venn diagrams with the number of students in the set represented by that region.