Products of 3rd grade multiplicative thinking and reasoning

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Introduction

The Common Core State Standards (National Governor's Association for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) are shifting education in a new direction- from the memorization of facts to the solving and reflecting upon more meaningful mathematical tasks that require deep Conceptual Understanding. Students are now expected to extend their thinking of each math problem by explaining their processes through words or diagrams (NGA & CCSSO, 2010). By requiring students to show their thinking, educators are able to see where students may struggle and further the learning of all students.

Mastery in multiplicative reasoning is now expected to start in 3rd grade and flow into 4th and 5th grade (NGA & CCSSO, 2010). Multiplication is a challenging concept for many students to fully gasp in depth. Multiplicative reasoning skills are crucial to students as they will use them throughout their life and as they continue to grow their mathematical understanding (National Council of Teachers of Mathematics [NCTM], 2000). Tasks that involve multiplicative reasoning involve a wide variety of strategies and as a result, more time (across grades 3-5) needs to be dedicated to it in the curriculum for students to be engaged meaningfully (NCTM, 2000).

Through our study we wanted to acknowledge the change that the Common Core Standards bring and the movement toward learning that is student driven. Our goal was to gain a better understanding of students' thought processes when approaching multiplication problems and to help their thinking develop. The guiding research question for our study was:

How can students' mathematical proficiency be developed in regard to multiplicative thinking and reasoning?

References
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
National Governor's Association for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Theoretical framework

The Common Core State Standards writing team (2011) outlined **how a student's learning progresses** when introduced to the concept of multiplication. There are three major types of problem situations in multiplication; in Grade 3 the focus is on two: equal groups of objects and arrays of objects. The team further noted that multiplication problem representations and solutions can be categorized into three levels (pp 25-26):

- **Level 1:** Students are counting and/or representing the entire amount in the multiplication task.
- **Level 2:** Students are able to use skip counting to solve their task.
- **Level 3:** Students are using higher level multiplicative properties to create and break down problems.

Conceptual Understanding	• Comprehension of mathematical concepts, operations, and relations
Procedural Fluency	• Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
Strategic Competence	• Ability to formulate, represent, and solve mathematical problems
Adaptive Reasoning	• Capacity for logical thought, reflection, explanation, and justification
Productive Disposition	•Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

The **five strands of mathematical proficiency**, as outlined by Kilpatrick, Swafford, and Findell (2001), refer to what is needed for a learner to fully develop mathematical thinking. It is important to recognize that all five strands are interdependent and are connected with one another. These strands are listed to the left from page 116 of *Adding it up: Helping Children Learn Mathematics.*

During the research we referenced a variety of educational articles specific to multiplication learning and instruction. The article *Teaching for Mastery in Multiplication* (Wallace & Guganus, 2005) describes meaningful teaching and learning methods of multiplication. The authors present important arguments as to why teaching with meaningful ideas and scenarios helps children build connections between concepts that are beneficial both for understanding the concepts and performing well on standardized tests. The authors suggest including hands on manipulatives as well as other forms of representations to solve problems. One of the methods employed by the authors is repeated addition, which uses groups of items, and explains multiplication as the total amount of the items in each group.

The article *Direct Modeling and Invented Procedures. Building on Students' Informal Strategies*, (Chambers, 1996) proposes the use of natural methods that young children can use to solve mathematical problems. Direct modeling of the problem situation provides a means through which students can begin to understand multiplication. Direct modelers use physical objects to act out story problems and to reach to an answer. Invented strategies can ultimately replace direct modeling. These are student-invented algorithms that reveal how students are making sense of a given problem.

References
Chambers, D. L. (1996). Direct modeling and invented procedures: Building on students' informal strategies. *Teaching Children Mathematics*, *3*(2), 92-95.
Common Core Standards Writing Team. (2011). *Progression for the common core state standards for mathematics (draft), K–5, operations and algebraic thinking*. Retrieved from http://commoncoretools.files.wordpress.com/2011/05/ccss_progression_cc_oa_k5_2011_05_302.pdf
Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
Wallace, A. H., & Gurganus, S. P. (2005). Teaching for mastery of multiplication. *Teaching Children Mathematics*, *12*(1), 26.

Methodology – Participants and procedure

Student Population

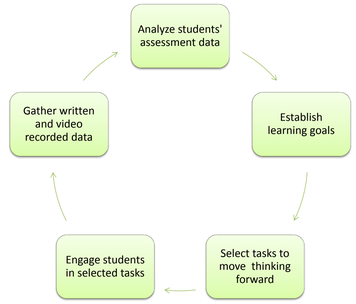
Grade level – students finishing 3rd grade
Number of participants – 4 students
Gender – 2 girls, 2 boys
Pseudonyms of participants – Tessa, Gabbie, Jake, Earl
Participation rate – 100% for 3 of the students, 1 student missed 2 lessons and post test
Duration of instruction – 7 1-hour instructional sessions
Pre and post assessment of 30 minutes clinical interview and 30 minutes written assessment

Common Core State Standards for Mathematics - Operations and Algebraic Thinking

CCSS.MATH.CONTENT.3.OA.A.1 - Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each.
CCSS.MATH.CONTENT.3.OA.A.3 - Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.
CCSS.MATH.CONTENT.3.OA.A.4 - Determine the unknown whole number in a multiplication or division equation relating three whole numbers.
CCSS.MATH.CONTENT.3.OA.B.5 - Apply properties of operations as strategies to multiply and divide.

National Governor's Association for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Retrieved from http://www.corestandards.org/.

PATHWAYS Cycle of Integrated Teaching and Research



The diagram shows the process we went through each week. During the first week we gathered both written and video recorded data which helped us establish the students' mathematical proficiency at that time. After that week we continued following the cycle of integrated teaching and research. Keeping in mind that our goal for each week was to move students' thinking forward in their mathematical proficiency, we selected tasks to assist them in achieving this goal.

Methodology – Data gathering and analysis

Pre and Post Interview Protocol

Students started by taking a **written assessment** which included a broad range of questions addressing third grade Common Core Standards. The students completed a 30-minute written assessment individually and then a 30-minute individual **clinical interview**. During the clinical interviews we aimed to learn about the students' thinking in terms of the five strands of mathematical proficiency. During the clinical interviews we strived to remain neutral to the students' responses and encouraged them to explain their mathematical thinking. We probed the students with follow-up questions to understand more about the process they use when approaching problems. Some key interview tasks were:

Ten rows of snails. Four snails in each row. How many snails?
The problem suggests using equal groups of objects and use of arrays. Students have to find the unknown product.

There are four boxes of crayons. Each box has 10 crayons in it. How many total crayons are there?
The problem from the clinical interview followed directly the snail problem. Students could make the connection and use a similar strategy.

There are 3 tables in Mrs. Potter's art classroom. There are 2 students sitting at each table. Each student has a box of 5 colored pencils.
How many colored pencils are at each table?
How many colored pencils do Mrs. Potter's students have in total?
Students had to attempt solving this multi step problem during the clinical interview. It involved finding the product by using equal groups of objects.

8 equal rows of cans, 48 total cans. How many cans in each row?
This problem from the written assessment describes a multiplication situation suggesting the use of arrays of objects. There is an unknown factor.

Procedures used in Research

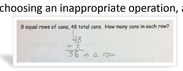
Video Recording – each instructional session as well as the clinical interviews were video recorded. We attempted to capture the students' facial expressions, and movements if the instruction involved such. In most cases we had two cameras running to make sure that every part of the conversation was audible.
Transcribing – Following each session we reviewed the video recording and created a written transcript of the conversation during the lesson. We also noted all the different emotions and movement throughout the video.
Analyzing the interviews – We went through the transcripts to look for evidence of any of the five strands of Mathematical proficiency that were displayed. Then we summarized each week's finding in a file which included strengths and weaknesses for each strand, conjectures to help students develop along each strand, as well as plans for the following lesson based on the findings.
Lessons – Each lesson was created to address one or more of the Common Core Standards, as described earlier. During the data summary we would reassess whether the lesson kept its goals towards the proper level of learning progressions. That would be our base on how to create the following week's lesson instructions and activities.
Student work samples – During each lesson we saved all of the students' written work and then scanned it for easier electronic access. Each work sample was also analyzed using the five strands of mathematical proficiency model.

Empirical Teaching and Learning Trajectory:

Initial Assessment Results

From the initial clinical interview and written assessment we were able to gauge where the students were with their approach to the variety of problems they completed. We noticed that there was a wide range of student understanding of multiplication coming into the program. Below are key examples of student work and thinking from the initial interviews.

- Earl and Gabbie had weakness in ***Conceptual Understanding***; They were not familiar with multiplication and wanted to turn most problems into addition problems when not appropriate. Students also demonstrated limited ***Strategic Competence*** in solving problems, sometimes choosing an inappropriate operation, as shown in Gabbie's work below:



- Jake showed strengths in his ***Conceptual Understanding*** of multiplication. He was consistently able to recognize situations that could be represented with multiplication and construct corresponding number sentences when appropriate. Jake also exhibited ***Procedural Fluency*** in computing multiplication and division results, as shown in the work sample below.

- Gabbie showed a need to develop ***Strategic Competence*** and ***Procedural Fluency*** in counting groups. In the snail problem (Shown in top right portion of poster) she had to count individually, and also made some counting errors.
- Some students were able to use drawings to represent questions which showed strength in their strategic competence. Jake drew arrays and pictures that showed equal groups for multiple questions. Below is an example of his equal groups:

- Earl showed strength in this area by using tiles to manipulate a problem that talked about 10 rows of 4 snails in each. He was able to organize them into groups to come to his final answer.
- Gabbie showed some weakness in this area through her struggle in approaching problems. Her lack of self-confidence played a role in limiting her ***Productive Disposition*** in approaching the problems and try to solve them in different perspectives.

Instructional cluster 1

The first two lessons involved problem situations about working with equal groups of objects and built on the students' understanding of repeated addition. We designed the tasks in a way that supported inquiry based learning and used physical manipulatives that the students could use to make sense of the problem. An attempt was made to connect the direct physical modeling of the problem situations into the relevant repeated addition and multiplication number sentences. The lessons were mostly geared towards ***Conceptual Understanding*** *and* ***Procedural Fluency*** mastery.

Lesson One – Repeating Patterns

- Students had to use a pattern to make a necklace and find out the total number of beads for each color if the pattern was repeated a certain amount of times. Below is an excerpt of Earl's work.

  - Gabbie was starting to grasp the concept of adding equal size groups, however in order to obtain the total amount she needed to count the beads individually
  - Tessa was able to easily identify the total number of beads for one color, but would change the number and size of the groups doing. In the problem shown above where the pattern is repeated three times, Tessa could identify that she would have a total of 6 yellow beads but she would explain that as 3+3.
  - Jake used his excellent recall of multiplication facts

Lesson Two – Word problems

- Covered a variety of word problems that dealt with having equal groups of objects and finding out how many total objects. A few example are included below.
  - The students acknowledged that skip counting would be a faster way to count. The example below was in regards to this problem, "Mr. Lee told 4 of his players to get 3 balls each and to start practicing":
    Silviya Gallo: You could count them by one or could we have a faster way?
    Earl: 3!
    SG: Could we count by 3? (Kids nod)
  - Tessa drew a rectangular array representation for the following problem. *"There are 4 gymnasts that are competing in a competition. Each gymnast has to include 5 cartwheels in their routine. How many cartwheels will the gymnasts perform altogether?"* In her repeated addition number sentence she explained it as 4 groups of 5 cartwheels.
  - Jake initially solved the above problem by using 5 groups of 4 cartwheels. But after further probing he changed his answer to match Tessa's response. This is what motivated us to move towards exploring the commutative property

Instructional cluster 2

In order to encourage the students' transition to **Level 2** in the learning progression we had them play a game that required skip counting along. The lessons allowed the student to build on their strategic competence through pattern seeking during the dialogue. It also strengthened their procedural fluency, as they expanded on the numbers they were counting by.

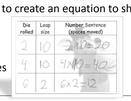
Lesson Three – Floor Game

- We started the lesson with an introductory word problem which helped bring out the commutative property of multiplication: "There is a special edition of Super Mario where you can play with 4 characters at the same time. To start the game each character has to jump 3 times. How many total jumps will there be before the game starts?" Students solved the problem in two different ways: (1) on own they all decided on 3+3+3+3=12; (2) by adding an additional condition students had to switch to 4+4+4=12.
- Then we switched gears with a floor game keeping the theme from the starting problem. There were 2 characters – Mario who would move by 5, and Yoshi who would move by 2. Students would roll a die which showed them how many times to count by 2 or by 5 depending on their character. The game was a very natural way to prompt skip counting, and to show it as a more efficient way than single counting. Struggles during the game indicated that the students may not be at Level 2 yet, and may benefit from additional work within the game context.
  - Gabbie needed assistance when counting by 2.
  - Both Tessa and Jake moved the wrong amount of spaces.
  - Jake showed fluency when counting by 5s and also identified a pattern of the numbers
    Mentor: Let's try it… 5… 10…
    Jake: 15, 20, 25, 30, 35… 40 […] You are going every 5 up.
    Mentor: Every 5 up. What do the numbers end in? 5, 10, 15, 20…
    Jake: 5 or 0.

Lesson Four – Board Game

- Continued the idea from the previous lesson, but we moved the game on the table to make it easier for the children to keep track and used different characters which had to move by 2s, 5s, 10s, 6s, 3s, and 4s. We asked the students to create an equation to show the spaces moved each time. We also asked them to create an equation to show their location on the board.

The example below is a small excerpt of Gabbie's work which shows how she was able to successfully record the number sentence representing the spaces she moved on the board game.



Instructional cluster 3

This cluster of activities focused on array representations of multiplication, and started introducing the concept of division. They promoted moving along the learning progressions by more exclusive use of multiplication equations and more abstract methods like arrays.

Lesson Five – "One Hundred Hungry Ants"

- During this activity we read parts of the 100 Hungry Ants book and used cut out squares of ants to build arrays to represent the marching ants. The arrays ranged from 1x100, 2x50, 4x25, 5x20, and 10x10 and the children discussed the corresponding number sentences. Students to experimented with whether the arrays could be with 3 lines or 6 or 7 lines.

Lesson Six – Array representations of 24

- We started with a group activity where the students were arranging their 24 objects in an array that they created for homework. Then the students were given a sheet of graph paper to cut out different arrays for the 24 creatures and then write the corresponding number sentences. At the end we displayed all the arrays and had a discussion about patterns we saw in all the arrays. This activity was aimed at hinting towards the idea of division, and it allowed the students to visually observe and understand the commutative property.

Gabbie did not write an equation in the assignment, but during the lesson was able to quickly reference the arrays she had created in her book

Earl could explain his reasoning for the array and number sentence. During the pattern seeking Earl could easily connect the array of 3 lines 8 with the array of 8 lines of 3 and demonstrated commutative property by turning the array sideways.

Jake showed flexibility in his ability to explain how to create an array moving in a different direction. All students initially started working at forming all of their 24 creatures in 1 line.
Jake: Put them in a sideways line.
NH: In a sideways line?
Jake: All the way across. 24 [lines] (pointing sideways).

Lesson Seven – Division problems and review

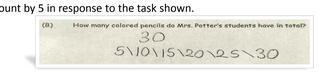
- Students had 4 different word problems that used the arrays created during the last session. The questions were structured in ways that promoted the ideas of division. Students related each problem to a multiplication and division sentence. At the end of the session we discussed different strategies that we could use to solve multiplication sentences if we didn't immediately know the answer.

During this lesson cluster we observed the students developing more ***Strategic Competence*** by employing new strategies to solve familiar tasks. They were also able to solve a new type of problem, including division, or finding an unknown product. During these instructional sessions the children were once again asked to search for patterns and make connections with their previous knowledge. We were able to observe more ***Adaptive Reasoning*** strengths.

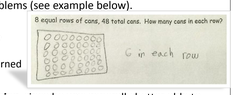
Post-Assessment Results

Following the completion of all the lessons and the final interviews we compared the students work from the pre- and post-assessments. We observed student growth in a variety of areas.

- Jake, who had demonstrated thorough understanding of multiplication at the beginning, completed the post assessment using more repeated addition to show his work. This showed growth in his ***Conceptual Understanding*** of the relationship between repeated addition and multiplication.
- Jake, Gabbie, and Earl all developed ***Procedural Fluency*** in skip counting. In the final interview Gabbie showed Level 2 thinking as she transitioned from individually counting the numbers to skip counting by 10, unprompted by the interviewer. Earl was able to skip count by 5 in response to the task shown.



- Gabbie demonstrated growth in ***Strategic Competence*** by acquiring knowledge of new mathematical representations. In the post assessments she was able to use a newly learned array representation to solve problems (see example below).
- Gabbie demonstrated growth in ***Strategic Competence*** by acquiring knowledge of new mathematical representations. In the post assessments she was able to use a newly learned array representation to solve problems



- Earl was able to develop ***Adaptive Reasoning***, since he was generally better able to explain his mathematical reasoning. The excerpt below is Earl explaining how he transitioned between 72 divided by 9 to his answer of 8x9=72.
  Earl: I counted all my 9s. Every single time I did it, I went back and counted so… and then when I had all… When I had 8 9s I… I did the last answer and I got the right answer.
  Silviya Gallo: That is a very interesting strategy. So… what does it say on here? What equation did you switch that to?
  E: I switched it to 8x9=72.
  SG: And how come you… used times?
  E: because… Times… um… doubled 9 8 times. It… repeated 9 8 times.
- Gabbie demonstrated growth in her ***Productive Disposition***, as she became more confident in approaching unfamiliar math problems. She went from saying "I don't know" to attempting to use skills she has developed to try and solve the problem.
- Jake is strong in his ***Adaptive Reasoning***; he generally enjoyed explaining the process he went through or how he viewed the problem.

Reflection and discussion:

Through this research we have developed tasks that are aligned with the Common Core Standards for 3rd grade multiplication. After having reflected on our work with the students we have found several of the standards more challenging to meet than others. One of the standards was 3.OA.A.4, specifically when students are given the product and have to find the unknown factor. This was challenging to have the students grasp because they were not familiar with problems that gave the product to start and took on the appearance of more of a division problem. Another standard that was difficult to fully meet was 3.OA.B.5, we worked purely on the commutative property for this standard. We dealt with the commutative property in multiple lessons but it is hard to gauge whether the students would be able to explicitly use this property without any probing questions, since they did not use it spontaneously to solve problems at any point

The learning progressions were a good base to measure where the students were with the concepts. Most students were able to move into **Level Two** methods by actively using their skip counting skills to group items. However, we found it to be a large step between **Level Two** and **Level Three** methods. It would be a smoother learning progression to have additional levels between the two levels as of now. In between the current levels it would be helpful to emphasize the patterns that are throughout multiplication and skip counting. Patterns such as when you are counting by 5 and the numbers end in 5 and 0. From our experience it would be helpful to add in the commutative property to the learning progressions prior to the **Level Three** methods with the higher multiplicative properties. The commutative property should be dealt with before **Level Three** because it can directly be connected to the array model. This property relates the two number sentences which students need to understand before composing and decomposing number sentences in **Level Three**.

It was helpful to use visuals and manipulatives even when students were at higher levels of the learning progressions. Creating realistic and relatable problems for the students to interact with also helped them move through the learning progressions. Problems that seemed to help students the most were those that involved active learning with hands on portions of the activities and problems that had multiple entry points to solve the problem with group discussions after.