The Thirty-Sixth Annual
Eastern Shore High School Mathematics Competition

November 14, 2019

Individual Contest Exam

Instructions

There are twenty problems on this exam. Select the best answer for each problem.

Your score will be the number of *correct* answers that you select.

There is no penalty for incorrect answers.

The use of a calculator is **not** permitted on this exam.

No computational work is required for any of your multiple choice responses.

However, in the event of tie scores, after the multiple choice responses have been checked for problems 1-20, the responses and/or written computational work on the enclosed form for problems #18, #19 and #20 will then be used as tiebreakers.

1. Solve $\dfrac{e^x + e^{-x}}{2} = 1$.

    a. no solution   b. $x = 0$ c. $x = 1$ d. $x = 0, 1$ e. $x = \ln(2)$

2. If $x = \sqrt[5]{-37}$, then which of the following must be true?

    a. $\sqrt{-x} > 2$
    b. $x > -2$
    c. $x^2 < 4$
    d. $x^3 < -8$
    e. $x^4 > 32$

3. An airplane flies 165 miles from point A in the direction $125°$ and then travels in the direction $245°$ for 80 miles. Approximately how far is the airplane from A?

    a. 110 miles   b. 127 miles   c. 143 miles   d. 155 miles   e. 170 miles

4. Which pair of equations represents two successive vertical asymptotes of the graph of $f(x) = \cot(2x - \pi)$?

    a. $x = -\dfrac{\pi}{4}, \ x = \dfrac{\pi}{4}$

    b. $x = 0, \ x = \pi$

    c. $x = \dfrac{\pi}{2}, \ x = \pi$

    d. $x = 0, \ x = 2\pi$

    e. $x = 0, \ x = 1$

5. Consider the system of inequalities:

$$\begin{cases} x > 0 \\ y \geq 0 \\ y \leq 6 - x^2 \\ y \geq 2x - 2. \end{cases}$$

Find the equation that shares exactly one solution with the system above.

    a. $y = -x$
    b. $y = 2x + 2$
    c. $y = x^2$
    d. $y = 4x - 6$
    e. None of the above.

6. Which of the following is equivalent to:

$$\left(\sqrt[4]{3\sqrt{3} - 3\sqrt{2}} \right) \left(\sqrt{3 + \sqrt{6}} \right) \left(\sqrt[4]{\sqrt{3} - \sqrt{2}} \right) ?$$

    a. $\sqrt{15}$ b. $\sqrt{3}$ c. 3   d. $\sqrt[4]{45}$ e. $\sqrt[4]{9 - 3\sqrt{6}}$

7. A sample of 100 high school students is selected.

- Each of the students is either a Freshman or Junior
- Each of the students is a member of either the Math Club or the Computer Science Club
- None of the students is a member of both the Math Club and the Computer Science Club
- 42 students are members of the Math Club
- 62 students are Juniors
- 18 students are members of the Math Club and Freshmen

Randomly select one student from these 100 students. What is the probability that the selected student is a member of the Computer Science Club or a Freshman?

a. $\dfrac{20}{100}$ b. $\dfrac{345}{1000}$ c. $\dfrac{526}{1000}$ d. $\dfrac{76}{100}$ e. $\dfrac{96}{100}$

8. A sample of 100 high school students is selected.

- Each of the students is either a Freshman or Junior
- Each of the students is a member of either the Math Club or the Computer Science Club
- None of the students is a member of both the Math Club and the Computer Science Club
- 42 students are members of the Math Club
- 62 students are Juniors
- 18 students are members of the Math Club and Freshmen

Randomly select one student from these 100 students. If you know that the selected student is a Junior, what is the probability that the selected student is a member of the Math Club?

a. $\dfrac{24}{62}$ b. $\dfrac{39}{100}$ c. $\dfrac{571}{1000}$ d. $\dfrac{76}{100}$ e. $\dfrac{4}{5}$

9. Consider an arithmetic sequence $a_1, a_2, \ldots$ where the sum of the first 10 terms is 100 and the sum of the first 50 terms is 2500. What is a_{25}?

a. 26   b. 27   c. 49   d. 51   e. It cannot be determined from the given information

10. For each four-digit positive integer $n = abcd$ (where $abcd$ is the decimal representation of n), let $f(n) = a + 2b + 3c + 4d$. So, for example, $f(2019) = 2 + 2(0) + 3(1) + 4(9) = 41$.

For how many integers, x, between 2020 and 2100, is the equation $f(x) = 41$ true?

a. 0   b. 1   c. 2   d. 3   e. 4

11. The radius of a circle inscribed in an isosceles triangle with a base of 12 units is 3 units. Find the perimeter of the isosceles triangle.

a. 27 units   b. 32 units   c. 37 units   d. 42 units   e. 47 units

12. The product of the roots of the equation $3 \cdot 81^{\frac{1}{x}} - 10 \cdot 9^{\frac{1}{x}} + 3 = 0$ is

a. -4   b. -1   c. 0   d. 1   e. 3

13. The dimensions of rectangle $MATH$ are $2\sqrt{2}$ units and $\dfrac{15\sqrt{2}}{2}$ units. A marble is released from the midpoint of $\overline{MH}$ and strikes $\overline{MA}$ at an angle of $45°$ and is reflected across the room to the opposite wall and so on until it strikes $\overline{AT}$. Note: rectangle $MATH$ is not drawn to scale.

M   A

H   T
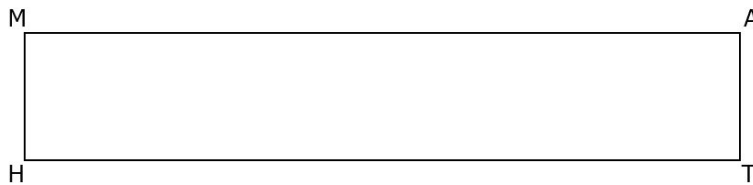
What is the length, L, of the marble's path when it strikes $\overline{AT}$?

a. 12 units $\leq L < 13$ units
b. 13 units $\leq L < 14$ units
c. 14 units $\leq L < 15$ units
d. 15 units $\leq L < 16$ units
e. 16 units $\leq L < 17$ units

14. When this block of Python code is executed, what is the output?

```
num = 4
total = 0
while num < 18:
        num = 2*num – 2
        total = total + num
print (total)
```

a. 0   b. 6   c. 16   d. 34   e. 68

15. Define a sequence a_n, such that the first term of the sequence is $a_1 = 5^3$, and the rest of the sequence is generated using the rule $a_n = 5^{7-4n}$, for $n \geq 2$.

The first three terms of a_n are 5^3, 5^{-1}, and 5^{-5}. What is the sum of all terms of this sequence?

a. $\dfrac{5^3}{5^4 - 1}$ b. $\dfrac{5^4}{5^4 - 1}$ c. $\dfrac{5^7}{5^4 - 1}$ d. 5^4 e. $\dfrac{5^{12}}{5^4 - 1}$

16. A randomly-chosen 20-year-old man has an 89% chance to live until his 50th birthday. For a randomly-chosen 20-year-old woman, this chance is 96%. Suppose a random pair of a 20-year-old man and 20-year-old woman get married. What is the probability that only one of them will be alive on their 50th birthday?

a. 0.036   b. 0.106   c. 0.141   d. 0.996   e. 1.850

(The answers have been rounded to three decimal places.)

17. In this problem, U is a set and for any subset X of U, X' represents the complement of X. In the figure below, what does the shaded region represent?
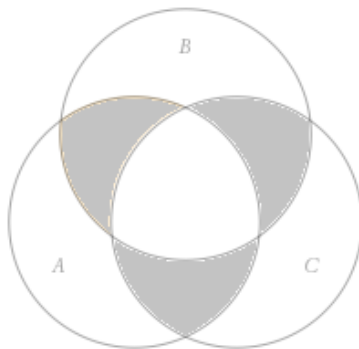
   a. $A \cap B \cap C$
   b. $(A \cap B \cap C)'$
   c. $(A \cap B) \cup (A \cap C) \cup (B \cap C)$
   d. $(A \cup B) \cap (A \cup C) \cap (B \cup C)$
   e. $(A \cap B \cap C') \cup (A \cap C \cap B') \cup (B \cap C \cap A')$

18. How many positive integers less than 2019 are a multiple of 20 or 19?

   a. 201   b. 202   c. 203   d. 204   e. 205

19. Consider an infinite stack of bricks, in which the bottom brick has a mass of 100 grams, the brick on top of it has a mass of 20 grams, the brick on top of the second has a mass of 4 grams, and each successive brick has a mass $\frac{1}{5}$ of the brick under it. What is the total mass of all these bricks?

   a. 120   b. 125   c. 130   d. 135   e. 140

20. A middle school student forgot her locker key password. The only thing she remembered was that none of the four digits (0 through 9) were repeated. To find out the code, she decided to try all possible arrangements of 4-digit codes. Assume it takes exactly 4 seconds to try each code. If the student took the maximum amount of time to decode the lock, the number of hours, H, until the lock was decoded satisfies which of the following?

   a. $H < 4.5$ b. $4.5 \leq H < 5.5$ c. $5.5 \leq H < 6.5$ d. $6.5 \leq H < 7.5$ e. $H \geq 7.5$