The Thirty-Second Annual
Eastern Shore High School Mathematics Competition

November 5, 2015

Team Contest Exam Solutions

1. Only 1 draw is needed before each cup can be labeled correctly PROVIDED the mislabeled cup "AB" is chosen FIRST.

   - Case 1:

     Assume the correctly labeled cups are mislabeled as shown.

     $$\begin{array}{ll} \text{Correct Labeling:} & \textbf{AA, AB, BB} \\ \text{Incorrect Labeling:} & \textbf{AB, BB, AA} \end{array}$$

     i.e., correctly labeled cup **AA** is mislabeled **AB**, correctly labeled cup **AB** is mislabeled **BB** and correctly labeled cup **BB** is mislabeled **AA**.

     After color **A** is drawn from the mislabeled cup **AB**, the cup can be correctly labeled **AA**. It then follows that mislabeled cups **BB** and **AA** can be respectively labeled correctly as **AB** and **BB**.

   - Case 2:
   Assume the correctly labeled cups are mislabeled as shown

     $$\begin{array}{ll} \text{Correct Labeling:} & \textbf{BB, AB, AA} \\ \text{Incorrect Labeling:} & \textbf{AB, AA, BB} \end{array}$$

     i.e., correctly labeled cup **BB** is mislabeled **AB**, correctly labeled cup **AB** is mislabeled **AA** and correctly labeled cup **AA** is mislabeled **BB**.

     After color **B** is drawn from the mislabeled cup **AB**, the cup can be correctly labeled **BB**. It then follows that mislabeled cups **AA** and **BB** can be respectively labeled correctly as **AB** and **AA**.

2. a. We can calculate the answer using a tree diagram:
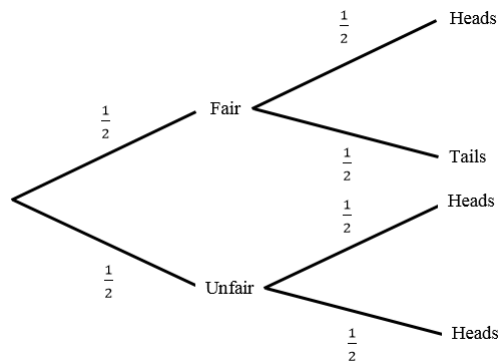
The two branches for each coin represent the two sides of each coin. For part a, since tails was not flipped, we must be in one of the three equally likely Heads branches. Only one of them corresponds to the fair coin, so the probability that the coin is fair is $\frac{1}{3}$. A similar diagram can be constructed for part b, and the diagrams can be generalized to answer part d.

b. Again we can calculate the answer using a tree diagram:
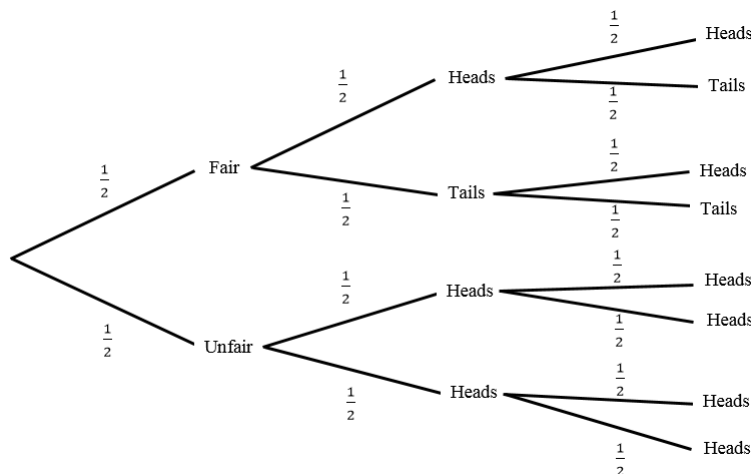
Since no tails were flipped, we must be in one of five equally likely Heads branches: Heads/Heads from the fair coin, or one of the four Heads/Heads branches from the unfair coin. Only one of them corresponds to the fair coin, so the probability that the coin is fair is $\frac{1}{5}$.

c. Since we obtained a flip of tails, the coin must be the fair coin, so the answer is 1.

d. We can generalize the reasoning from parts a and b to find that the probability is $\frac{1}{1+2^n}$.

3. a. If only two consecutive page numbers are circled, one must be even and the other odd. The sum of an even number and an odd number is odd, so s must be odd. Therefore if s is even, the number of circled pages cannot be 2.

b. If s is an odd number greater than 1, then $s = 2k + 1$ for some positive integer k. A solution in this case is that 2 consecutive page numbers are circled, k and $k + 1$.

c. Let k be the number of circled page numbers and n the smallest circled page number. Then,

$$s = n + (n + 1) + \cdots + (n + (k - 1)) = kn + (1 + \cdots + (k - 1))$$

Using the formula for the sum of the first $k - 1$ whole numbers, we can rewrite s as

$$s = kn + \frac{(k - 1)k}{2}$$

This implies that $k(2n + k - 1) = 2s$. (**)

Since $s = 500$, we have $k(2n + k - 1) = 1000$, so k must be a factor of 1000. Since we seek a solution other than the trivial solution, $k \neq 1$, and from part a, $k \neq 2$. If $k = 5$, then $2n + k - 1 = 200$ which implies $n = 98$. One solution then is that page numbers 98, 99, 100, 101, and 102 are circled. When $k = 8$, we have that page numbers 59 through 66 are circled.

Notice that if k is a factor of 1000 larger than 8, the equation (**) implies that n is negative. Thus, we have found the only two solutions (other than the trivial solution) when $s = 500$.

d. From part c, we see that when $s = 500$, there are two solutions other than the trivial solution.

Other such values of s can be found by choosing s to be an odd triangular number. For example, when $s = 55$, two solutions are that (1) page numbers 27 and 28 are circled and (2) page numbers 1 through 10 are circled.