1. Your team has been provided with a set of snap cubes. For the following instructions, consider each snap cube to be one cubic unit - that is, each snap cube measures 1 unit by 1 unit by 1 unit (or just "$1 \times 1 \times 1$," for short). For example, a cube that is constructed using eight snap cubes would measure 2 units by 2 units by 2 units (or "$2 \times 2 \times 2$").

   (a) Use eight snap cubes to construct a $2 \times 2 \times 2$ cube. If we were to *paint* the exterior of this $2 \times 2 \times 2$ cube, what would be the correct answer to each of the following questions?

      i. How many of the snap cubes have paint on exactly one side?
      ii. How many of the snap cubes have paint on exactly two sides?
      iii. How many of the snap cubes have paint on exactly three sides?
      iv. How many of the snap cubes have paint on more than three sides?

   (b) Use 27 snap cubes to construct a $3 \times 3 \times 3$ cube. If we were to paint the exterior of this $3 \times 3 \times 3$ cube, what would be the correct answer to each of the following questions?

      i. How many of the snap cubes have paint on exactly one side?
      ii. How many of the snap cubes have paint on exactly two sides?
      iii. How many of the snap cubes have paint on exactly three sides?
      iv. How many of the snap cubes have paint on more than three sides?

   (c) Use 64 snap cubes to construct a $4 \times 4 \times 4$ cube. If we were to paint the exterior of this $4 \times 4 \times 4$ cube, what would be the correct answer to each of the following questions?

      i. How many of the snap cubes have paint on exactly one side?
      ii. How many of the snap cubes have paint on exactly two sides?
      iii. How many of the snap cubes have paint on exactly three sides?
      iv. How many of the snap cubes have paint on more than three sides?

   (d) Now that you've got the idea - imagine using $n^3$ snap cubes to construct an $n \times n \times n$ cube (where $n$ may be any counting number). If we were to paint the exterior of this $n \times n \times n$ cube, what would be the correct answer (in terms of $n$) to each of the following questions?

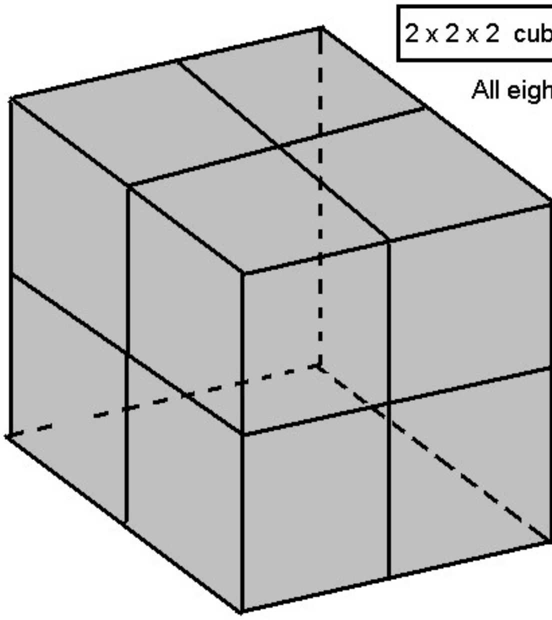
      i. How many of the snap cubes have paint on exactly one side?
      ii. How many of the snap cubes have paint on exactly two sides?
      iii. How many of the snap cubes have paint on exactly three sides?
      iv. How many of the snap cubes have paint on more than three sides?

Solutions:

(a)

2 x 2 x 2 cube

All eight cubes are painted on three sides.



(b)

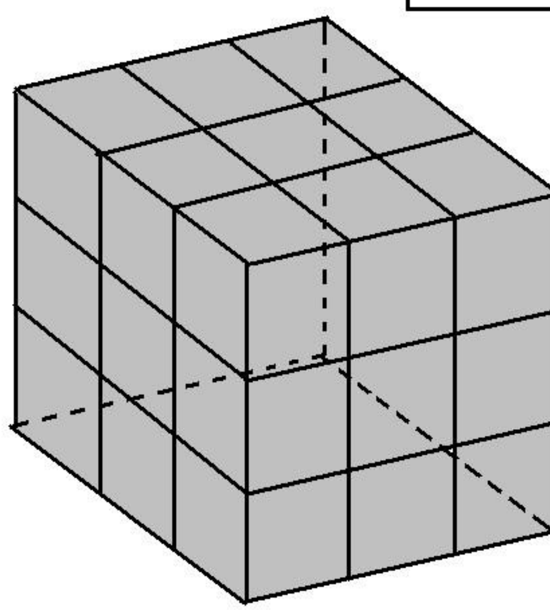
3 x 3 x 3 cube

i) Six snap cubes (one in the center of each face of the large cube) are painted on exactly one side.

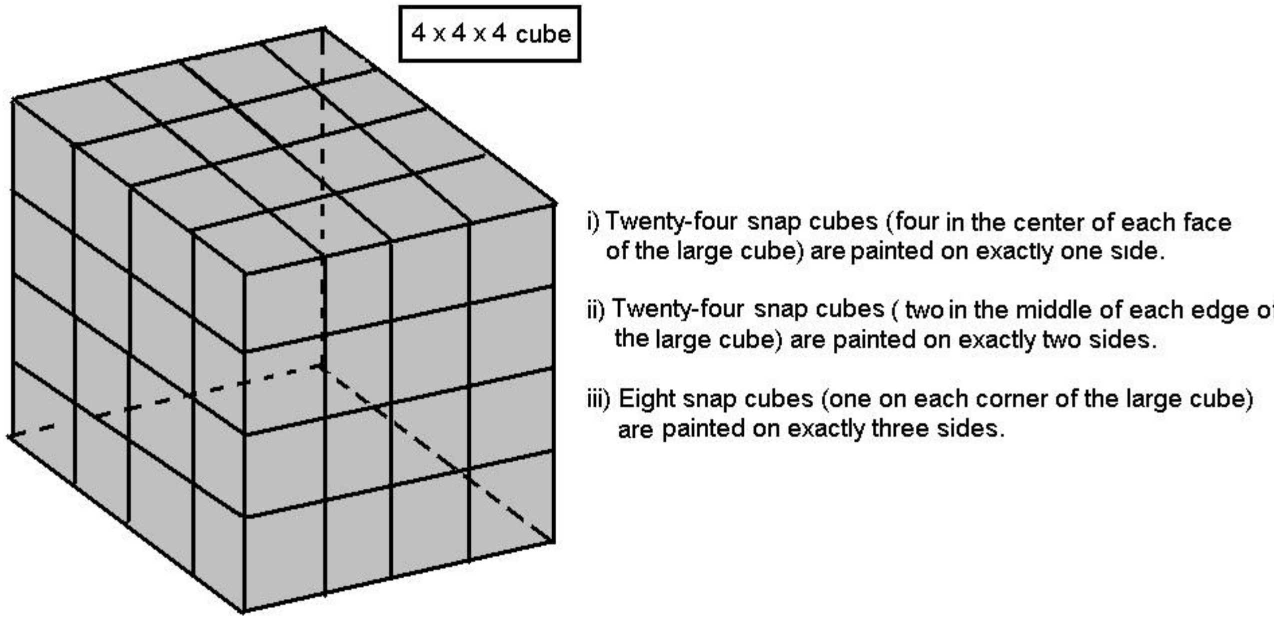
ii) Twelve snap cubes (one in the middle of each edge of the large cube) are painted on exactly two sides.

iii) Eight snap cubes (one on each corner of the large cube) are painted on exactly three sides.

iv) Zero snap cubes are painted on more than three sides. (Note: the answer to part (iv) is always zero, regardless of the size of the large cube.)



(c)




(d) For an $n \times n \times n$ cube, the results would be as follows:

i. There would be $(n-2)^2$ snap cubes on each face of the cube painted on exactly one side, so the answer here is $6(n-2)^2$ snap cubes.

ii. There would be $n-2$ snap cubes on each edge of the cube painted on exactly two sides, so the answer here is $12(n-2)$ snap cubes

iii. There would be 8 snap cubes - one on each corner of the cube - painted on exactly three sides.

iv. For all $n \geq 2$, there will be no snap cubes painted on more than three sides.

2. Solve for $x$ in terms of $a$:

$$2\left(\frac{x-1}{x-a}\right)^2 - 5\left(\frac{x-1}{x-a}\right) + 2 = 0.$$

Solution: First substitute $y = \dfrac{x-1}{x-1}$, so that we can rewrite the equation in the form $2y^2-5y+2=0$. This factors as $(2y-1)(y-2)=0$, so the solutions (in terms of $y$) are $y = 1/2, y = 2$. Now, solve for $x$:

$$\begin{aligned}
y &= \frac{x-1}{x-a} \\
y(x-a) &= x-1 \\
xy - ay &= x-1 \\
xy - x &= ay-1 \\
x(y-1) &= ay-1 \\
x &= \frac{ay-1}{y-1}
\end{aligned}$$

Thus,

$$y = \frac{1}{2} \quad \rightarrow \quad x = \frac{a/2 - 1}{-1/2} = 2 - a$$
$$y = 2 \quad \rightarrow \quad x = \frac{2a - 1}{2 - 1} = 2a - 1$$

Therefore, the solution set of the equation is $\{2 - a, 2a - 1\}$.

3. Mohammad's house is on a road where the house numbers run 1, 2, 3, 4, ..., consecutively. By a curious coincidence, the sum of all house numbers less than his house number is the same as the sum of all house numbers greater than his house number. Assuming Mohammad's house number is in the thirties, what is his house number, and how many houses are on his road?

   Solution: Let $x$ denote Mohammad's house number, and let $y$ denote the number of houses on the road. Then, we know $30 \leq x \leq 39$, and

$$1 + 2 + 3 + \ldots + (x-2) + (x-1) = (x+1) + (x+2) + \ldots + (y-1) + y.$$

   In general, the formula for the sum of the first $n$ positive integers is

$$1 + 2 + \ldots + n = \frac{(n+1)n}{2}.$$

   So, the left-hand side of the above equation can be rewritten in the form

$$\frac{(x-1)((x-1)+1)}{2} = \frac{x(x-1)}{2}.$$

   The right-hand side can be looked at as the difference between $1 + 2 + 3 + \ldots + y$ and $1 + 2 + 3 + \ldots + x$, which (using the same formula) can be written as $\frac{y(y+1)}{2} - \frac{x(x+1)}{2}$. Thus, we have

$$\begin{aligned}
1 + 2 + 3 + \ldots + (x-2) + (x-1) &= (x+1) + (x+2) + \ldots + (y-1) + y \\
1 + 2 + 3 + \ldots + (x-2) + (x-1) &= 1 + 2 + 3 + \ldots + y - (1 + 2 + 3 + \ldots + x) \\
\frac{x(x-1)}{2} &= \frac{(y+1)y}{2} - \frac{(x+1)x}{2} \\
x(x-1) &= y(y+1) - x(x+1) \\
x^2 - x &= y^2 + y - x^2 - x \\
2x^2 &= y^2 + y \\
y^2 + y - 2x^2 &= 0
\end{aligned}$$

   If we use the quadratic equation to solve this equation for $y$, we obtain

$$y = \frac{-1 \pm \sqrt{1 + 8x^2}}{2}.$$

   So, in order for $y$ to be an integer, it is required that $1 + 8x^2$ be a perfect square. Recall that $30 \leq x \leq 39$; the only integer $x$ in this range for which $1 + 8x^2$ is a perfect square is $x = 35$ (since $1 + 8 \cdot 35^2 = 9801 = 99^2$). So, $x = 35$, which implies

$$y = y = \frac{-1 + \sqrt{1 + 8 \cdot 35^2}}{2} = \frac{-1 + 99}{2} = \frac{98}{2} = 49.$$

Thus, Mohammad's house number is 35, and there are 49 houses on his road.

(Check:

$$1 + 2 + \ldots + 34 = \frac{34 \cdot 35}{2} = 595,$$

and

$$36 + \ldots + 49 = \frac{49 \cdot 50}{2} - \frac{35 \cdot 36}{2} = 1225 - 630 = 595.$$

4. (Loosely based on a true story.) On The Parkway, the toll at any exit can be paid with a token, regardless of the current toll charge or of how long ago the token was purchased.

Prior to 1989, the toll at each Parkway exit had been 25 cents; accordingly, the Parkway Authority charged \$10 for a roll of 40 tokens. On January 1, 1989, the Parkway Authority increased exit tolls from 25 cents to 35 cents, but continued to sell rolls of 40 tokens for \$10 per roll. At this time, they announced that on April 1, 1989, they would reduce the number of tokens per roll from 40 to 30 (while still charging \$10 per roll). Shortly after this announcement, there was a sudden increase in demand for tokens; as a result, the toll-takers, who sold the tokens, were frequently out of them.

Imagine that it is January 2, 1989, you are a person who drives on the Parkway regularly, and you have just learned about the planned price increase on tokens. Since tokens are cheaper now than they will be on April 1, you need to decide how many rolls of tokens to buy now at the lower price.

Assume that any money you do not spend on rolls of tokens could instead be invested in a savings account that will pay 4% interest compounded annually. Also, assume that each token you buy can be used forever – that is, tokens do not expire. Based on these assumptions, how many rolls of tokens should you buy?

(Hint: your answer may depend in part on information not given here. To account for this, you may want make a reasonable assumption, or - preferably - use a variable to represent an unknown quantity. Include this assumption and/or variable in your written work, and make it clear how this factors into your answer.)

Solution: When tokens go from \$10 for 40 to \$10 for 30, that represents an immediate 100/3% (or 33.3333...%) increase in "value," so what we need to determine is how long will it take for money in the bank at 4% compounded annually to earn 33.33...%. Thus we need to solve the equation $P_0(1.04)^n = P_0(1.333333\ldots)$ for $n$:

$$\begin{aligned}
P_0(1.04)^n &= P_0(1.333333\ldots) \\
(1.04)^n &= 1.333333\ldots \\
\log\left((1.04)^n\right) &= \log(1.333333\ldots) \\
n\log(1.04) &= \log(1.333333\ldots) \\
n &= \frac{\log(1.333333\ldots)}{\log(1.04)} \\
n &\approx 7.33
\end{aligned}$$

(Note: It doesn't matter which logarithm you use here, since the properties of logarithms hold for any base – as long as you use the SAME logarithm on both sides of the equation.)

So you would want to buy approximately 7 years and 4 months supply. (There is an assumption that the bank would give you proportionate interest on an investment for a partial year or that interest can begin to accrue at any time.) Assuming that you can estimate that on average you use $n$ tokens per month, that would be $(7(12)+4)n$ tokens, or $88n$ tokens. Of course you need to get tokens in rolls of 40 so you would probably choose to buy $\lfloor 2.2n \rfloor$ rolls (where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.)

Note: If the interest had been compounded *continuously*, rather than annually, then the solution would have been as follows:

$$\begin{aligned}
P_0 e^{0.04t} &= P_0(1.333333\ldots) \\
e^{0.04t} &= 1.333333\ldots \\
0.04t &= \ln(1.333333\ldots) \\
t &= \frac{\ln(1.333333\ldots)}{0.04} \\
n &\approx 7.19,
\end{aligned}$$

so you would purchase approximately 7 years and 2 month's supply, or $86n$ tokens...