2009 HSMC Team Contest Solutions

1. We will solve part (c) first, and use our results to solve (a) and (b).

   Bob's rate during the n^{th} hour of walking is $2 + k(n-1)$ km per hour. So, the rate at which Andy and Bob approach each other is $4+2+k(n-1) = 6+k(n-1)$ km per hour. Therefore...

   - After 1 hour, the distance between them is $30 - 6 = 24$ miles.
   - After 2 hours, the distance is $24 - (6+k) = 18 - k$ (as long as $k < 18$).
   - After 3 hours, the distance is $18 - k - (6+2k) = 12 - 3k$ (if $k < 4$).
   - After 4 hours, the distance is $12 - 3k - (6+3k) = 6 - 6k$ (if $k < 1$).
   - After 5 hours, the distance is $6 - 6k - (6+4k) = -10k$. That is, Andy and Bob pass either other within 5 hours if k is any positive number; they meet after exactly 5 hours if k is exactly zero.

   Thus, we have the following possibilities:

   - $1 \le k \le 4$: In this case, Andy and Bob meet sometime during the fourth hour. After three hours, the distance between them is $12 - 3k$ (as demonstrated above), and the relative speed at which they are approaching each other during the fourth hour is $6+3k$. So, the number of hours required to travel $12-3k$ km at a rate of $6+3k$ km per hour is $\frac{12-3k}{6+3k}$, which simplifies to $\frac{4-k}{2+k}$. Thus, if $0 < k < 1$, then it takes $3 + \frac{4-k}{2+k}$, or $\frac{10+2k}{2+k}$, hours for Andy and Bob to reach each other.

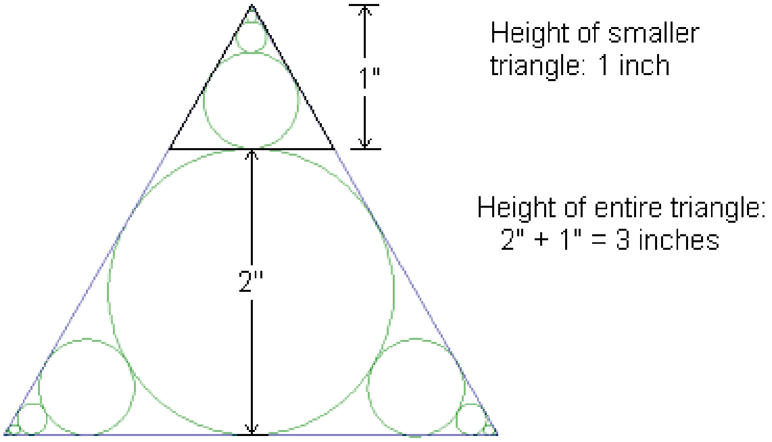
     Note: We can now answer (a). If $k = 1$, then it takes $3 + 3/3 = 4$ hours for Andy and Bob to reach each other.
   - $0 \le k \le 1$: In this case, Andy and Bob meet during the fifth hour. After four hours, the distance between them is $6 - 6k$, and the relative speed at which they are approaching each other during the fifth hour is $6 + 4k$. So, the number of hours required to travel $6 - 6k$ km at a rate of $6 + 4k$ km per hour is $\frac{6-6k}{6+4k}$, which simplifies to $\frac{3-3k}{3+2k}$. Thus, if $0 < k < 1$, then it takes $4 + \frac{3-3k}{3+2k}$, or $\frac{15+5k}{3+2k}$, hours for Andy and Bob to reach each other.

     Note: We can now answer (b). If $k = 1/2$, then it takes $4 + \frac{1.5}{4} = 4\frac{3}{8}$, or 4.375, hours for Andy and Bob to reach each other.

2. Clearly the circumference of the largest circle (with radius 1") is 2π inches. Now, we'll start by finding the radius of each of the three circles that are tangent to the largest circle. (Since all three of these will be congruent, just focus on one of them; say, the one at the "top," as oriented on the test sheet.)

   Draw a line that is parallel to the base of the equilateral triangle and tangent to both the larger circle and the smaller circle. This line will cut the equilateral triangle into two regions; the top region (containing the smaller circle) is again an equilateral triangle (as shown in the diagram below). The radius of a circle inscribed in an equilateral triangle is 1/3 of the height (or altitude) of that triangle. (To see why this is the case, sketch a 30-60-90 triangle with vertices at the center of the inscribed circle, one vertex of the equilateral triangle, and the midpoint of one side of the equilateral triangle.) Therefore, the diameter of the inscribed circle is 2/3 of the height of the triangle. Thus, when we divided the first equilateral triangle into two regions, the smaller region was in fact an eqilulateral triangle whose height is 1/3 of the height of the original triangle.

By geometric similarity, then, the circumference of the circle inscribed in this smaller equilateral triangle is 1/3 of that of the next-larger circle. This pattern continues indefinitely - that is, as we inscribe smaller and smaller circles closer to each vertex of the original equilateral triangle, each new circle's circumference is 1/3 of the previous circle's circumference.



So, the sum of the circumferences of the circles approaching the "top" of the equilateral triangle is:

$$2\pi(1/3) + 2\pi(1/3)^2 + 2\pi(1/3)^3 + \ldots$$

This simplifies to 2π times the sum of the geometric series $1/3 + (1/3)^2 + (1/3)^3 + \ldots$; that is, $2\pi \cdot \left(\frac{1/3}{1-1/3}\right) = \frac{2\pi}{2} = \pi$. Since there are three such sequences of circles (one approaching each vertex of the equilateral triangle), the sum of the circumferences of all of these circles is 3π. Adding the circumference of the largest circle (which isn't counted in any of these infinite sums), the sum of the circumferences of all of the inscribed circles is $3\pi + 2\pi = 5\pi$.

3. The answers, with brief explanations, are as follows.

   (a) False.

   There are 15+21+24=60 students who own a dog; 15 of these are only children. Therefore, a student who owns a dog has a $15/60 = 0.25$ probability of being an only child. Similarly, a student who does not own a dog has a $9/30 = 0.3$ probability of being an only child. Therefore, a student does not own a dog is *more* likely to be an only child than a student who does own a dog.

   (b) True.

   Of the 90 students, there are 21+24=45 who own a dog and have at least one sibling. Therefore, a randomly selected student from the class has a $45/90 = 0.5$ probability of owning a dog and having at least one sibling.

   (c) False.

   Of the 37 students with more than 2 siblings, 24 own dogs, so a student with more than two siblings has a $24/37 \approx 0.65$ probability of owning a dog. Of the 29 students with one or two siblings, 21 own a dog, so a student with 1 or 2 siblings has a $21/29 \approx 0.72$ probability of owning a dog. Therefore, a student with more than two siblings is *less* likely to own a dog than a student with one or two siblings.

4.

$$\begin{aligned}
&\left(\frac{1}{2a-b}+\frac{3b}{b^2-4a^2}-\frac{2}{2a+b}\right)\div\left(\frac{4a^2+b^2}{4a^2-b^2}+1\right)\\
&= \left(\frac{1(2a+b)}{(2a-b)(2a+b)}+\frac{-3b}{-(b-2a)(b+2a)}-\frac{2(2a-b)}{(2a-b)(2a+b)}\right)\div\left(\frac{4a^2+b^2+(4a^2-b^2)}{4a^2-b^2}\right)\\
&= \left(\frac{2a+b-3b-(4a-2b)}{(2a-b)(2a+b)}\right)\div\left(\frac{8a^2}{4a^2-b^2}\right)\\
&= \left(\frac{-2a}{4a^2-b^2}\right)\times\left(\frac{4a^2-b^2}{8a^2}\right)\\
&= \frac{-2a}{8a^2}\\
&= -\frac{1}{4a}
\end{aligned}$$

5. To answer the questions about S, we must first notice that there are 9!, or 362880, permutations of the digits 1 through 9, which implies there are 362880 numbers in S.

   (a) There are $8! \times 4$ numbers in S that end with 2, 4, 6 or 8. Therefore, the probability of selecting such a number at random is $\frac{8! \times 4}{9!} = \frac{4}{9}$, or about 0.444.

   (b) There are $7 \times 8!$ numbers in S whose first digit is 3, 4, 5, 6, 7, 8 or 9. Therefore, the probability of selecting such a number at random is $\frac{8! \times 7}{9!} = \frac{7}{9}$, or about 0.777.

   (c) There are $5 \times 4 \times 3 = 60$ ways to choose the first, fifth and ninth digits from among 1,3,5,7,9. After this selection is made, there are 6! ways to choose the other 6 digits of a 9-digit number. Therefore, there are $60 \times 6!$ numbers in S whose first, fifth and ninth digits are odd, and so the probability of selecting such a number at random is $\frac{6! \times 60}{9!} = \frac{60}{9 \times 8 \times 7} = \frac{5}{42} \approx 0.119$.

   (d) The possibilities are as follows:

   - Last digit 3; First two digits 12 (e.g. 124567893) or 21 (e.g. 214567893)
   - Last digit 4; First two digits 13 (e.g. 139876524) or 31 (e.g. 319876524)
   - Last digit 5; First two digits 14, 23, 32 or 41
   - Last digit 6; First two digits 15, 24, 42 or 51
   - Last digit 7; First two digits 16, 25, 34, 43, 52 or 61
   - Last digit 8; First two digits 17, 26, 35, 53, 62 or 71
   - Last digit 9; First two digits 18, 27, 36, 45, 54, 63, 72 or 81

   This gives us a total of 32 combinations of first two digits and last digit. For each of these, there are 6! ways to select the other six digits. Therefore, there are $32 \times 6!$ numbers in S satisfying this condition, and so the probability of selecting such a number at random is $\frac{32 \times 6!}{9!} = \frac{32}{9 \times 8 \times 7} = \frac{4}{63} \approx 0.063$.