

Eastern Shore High School Mathematics Competition
2008 Team Contest Solutions

1. (Manipulative-based problem - see the exam for the text of this problem.)

(a)

- i. The empirical probability depends entirely on your data. Your answer should be a fraction whose numerator is the number of successful trials and whose denominator is the total number of trials you ran. So, for example, if you had 7 successful trials and 25 unsuccessful trials, then your empirical probability should be $\frac{7}{7+25} = \frac{7}{32}$.
- ii. There are (at least) two valid methods for finding the theoretical probability of success.
 - Conditional probability: At each step, the probability of having the correct sequence of cards up to that point is equal to the probability of having the correct sequence up to the previous step, multiplied by the probability of selecting the correct card for the current step *given that the correct cards have already been selected up to that point*. To calculate the probability, then, you must keep a running count of how many cards total, and how many copies of the card you need to draw, are left at each step.

First card: You need to select a 5. There are two 5's among the four face-down cards, so your probability of drawing a 5 is $2/4$.

Second card: Assuming you selected a 5 as your first card, there are two 2's left among the remaining three cards, so your probability of drawing a 2 is $2/3$. Therefore, your probability of having the right sequence so far is $\frac{2}{4} \times \frac{2}{3} = \frac{1}{3}$.

Third card: Assuming you've selected the right sequence of cards so far, there are two cards left, one of which is the other 2 that you need to draw. So, your probability of selecting the 2 is $1/2$. Therefore, your probability of having the right sequence so far is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

Fourth card: Assuming you've selected the right sequence of cards so far, the only remaining card is a 5, so you have a 100% chance of selecting it. Therefore, the theoretical probability of selecting the correct sequence of cards (5, 2, 2, 5) is exactly $1/6$.

- Counting permutations: Imagine that the four cards are colored in such a way that there is one red 5, one blue 5, one red 2, and one blue 2. (The cards aren't really colored, but imagining them to be doesn't affect their likelihood of being selected, so this is mathematically valid.) Then, there are four distinct cards being selected in order. There are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ equally likely ways in which this selection can be made. Of these 24, there are exactly 4 ways in which we get the right sequence of 2's and 5's: $\{5_{red}, 2_{red}, 2_{blue}, 5_{blue}\}$, $\{5_{red}, 2_{blue}, 2_{red}, 5_{blue}\}$, $\{5_{blue}, 2_{red}, 2_{blue}, 5_{red}\}$, $\{5_{blue}, 2_{blue}, 2_{red}, 5_{red}\}$.

Thus, there are four (out of twenty-four equally likely) selections which give us the right sequence. Therefore, the probability of selecting the right sequence is $\frac{4}{24}$, or $\frac{1}{6}$.

(b)

- i. (See part (a))
- ii. Again, there are (at least) two valid methods for finding the theoretical probability of success.

- Conditional probability: Similarly to part (a):

First card: You need to select a 5. There are two 5's among the five face-down cards, so your probability of drawing a 5 is $2/5$.

Second card: Assuming you selected a 5 as your first card, there are two 2's left among the remaining four cards, so your probability of drawing a 2 is $2/4$.

Therefore, your probability of having the right sequence so far is $\frac{2}{5} \times \frac{2}{4} = \frac{1}{5}$.

Third card: Assuming you've selected the right sequence of cards so far, there are three cards left, one of which is the equals sign that you need to draw. So, your probability of selecting the right card now is $1/3$. Therefore, our probability of having the right sequence so far is $\frac{2}{5} \times \frac{2}{4} \times \frac{1}{3} = \frac{1}{15}$.

Fourth card: Assuming you've selected the right sequence of cards so far, there are two cards left, one of which is the other 2 that you need to draw. So, your probability of selecting the 2 is $1/2$. Therefore, your probability of having the right sequence so far is $\frac{2}{5} \times \frac{2}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{30}$.

Fifth card: Assuming you've selected the right sequence of cards so far, the only remaining card is a 5, so you have a 100% chance of selecting it. Therefore, the theoretical probability of selecting the correct sequence of cards (5, 2, =, 2, 5) is exactly $1/30$.

- Counting permutations: As in part (a), imagine that the five cards are colored in such a way that there is one red 5, one blue 5, one red 2, and one blue 2. Then, there are five distinct cards being selected in order. There are $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ equally likely ways in which this selection can be made. Of these 120, there are exactly 4 ways in which we get the right sequence of 2's and 5's: $\{5_{red}, 2_{red}, =, 2_{blue}, 5_{blue}\}$, $\{5_{red}, 2_{blue}, =, 2_{red}, 5_{blue}\}$, $\{5_{blue}, 2_{red}, =, 2_{blue}, 5_{red}\}$, $\{5_{blue}, 2_{blue}, =, 2_{red}, 5_{red}\}$.

Thus, there are four (out of 120 equally likely) selections which give us the right sequence. Therefore, the probability of selecting the right sequence is $\frac{4}{120}$, or $\frac{1}{30}$.

2. The quadrilateral ABCD (shown below) is a trapezoid, with \overleftrightarrow{AB} parallel to \overleftrightarrow{DC} . Points E and F are the midpoints of \overline{AD} and \overline{BC} , respectively. Line segments \overline{DB} and \overline{EF} intersect at P , and \overline{AC} and \overline{EF} intersect at Q .

(a) If $|AB| = 18$ and $|DC| = 28$, then what is the value of $\frac{|PQ|}{|EF|}$?

(b) More generally: if $|AB| = m$ and $|DC| = n$, where m, n are both positive and $m < n$, then then what is the value of $\frac{|PQ|}{|EF|}$, in terms of m and n ?

Solution: The answer to (b) is $\frac{n-m}{n+m}$.

It follows that the answer to (a) (substituting $m = 18$ and $n = 28$) is $\frac{10}{46}$, or $\frac{5}{23}$.

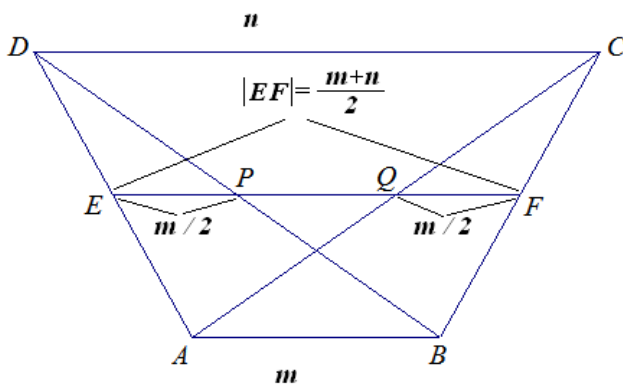
For the general solution, we observe that $\triangle DEP \sim \triangle DAP$, and so $|EP| = |AB|/2 = m/2$. Similarly, $|QF| = |AB|/2 = m/2$. Also, since \overline{EF} has its endpoints at the midpoints of \overline{AD} and \overline{BC} , its length is the average of the lengths of the trapezoid's bases, so $|EF| = \frac{m+n}{2}$.

Thus,

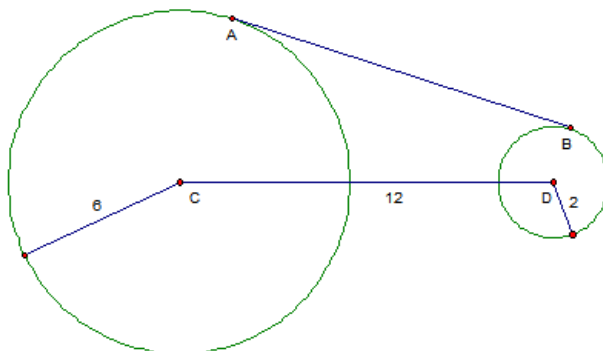
$$|PQ| = |EF| - |EP| - |QF| = \frac{m+n}{2} - \frac{2m}{2} = \frac{n-m}{2},$$

and so

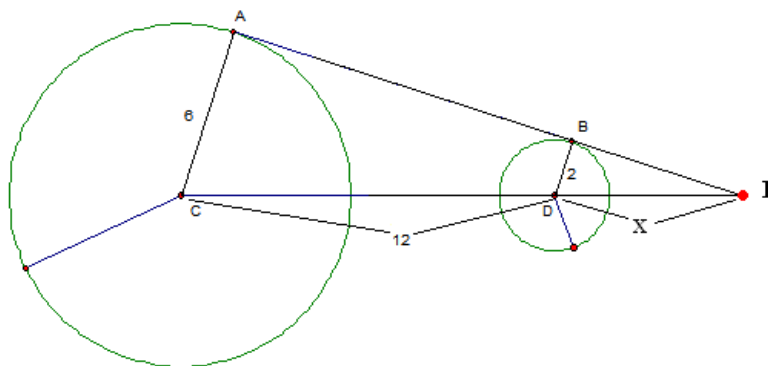
$$\frac{|PQ|}{|EF|} = \frac{\frac{n-m}{2}}{\frac{m+n}{2}} = \frac{n-m}{m+n}.$$



3. Two circles, with radii 6 and 2, are coplanar, and their centers are 12 units apart, as shown below. Line \overleftrightarrow{AB} is a common external tangent line to the two circles, where A and B are the points at which this tangent intersects the larger circle and the smaller circle, respectively. Find the length of line segment \overline{AB} .



The key to this problem is to extend lines \overleftrightarrow{AB} and \overleftrightarrow{CD} to find their intersection, which is labeled E in the diagram below.



We see that $\triangle ACE$ and $\triangle BDE$ are both right triangles, and $\triangle ACE \sim \triangle BDE$, with $|AC|/|BD| = 3$. Therefore, $|CE|/|DE| = 3$. If we let $x = |DE|$ (as shown in the diagram), then $|CE| = 12 + x$, so we have the equation $\frac{12+x}{x} = 3$, whose solution is $x = 6$. Therefore, $|DE| = 6$, and so $|CE| = 18$. Thus, \overline{AE} is one leg of a right triangle whose other leg, \overline{AC} , has length 6, and whose hypotenuse, \overline{CE} , has length 18. Therefore, $|AE| = \sqrt{18^2 - 6^2} = \sqrt{288}$, which simplifies as $|AE| = 12\sqrt{2}$. By the similarity between $\triangle ACE$ and $\triangle BDE$, we find $|BE| = |AE|/3 = 4\sqrt{2}$. Therefore, $|AB| = 12\sqrt{2} - 4\sqrt{2} = 8\sqrt{2}$.

4. Let $p = \log_4 3$ and $q = \log_3 5$. Find the *exact* value of the following in terms of p and q .

The following solutions make extensive use of properties of logarithms. For any step that you do not understand, you should consult a textbook to refresh your understanding of the many useful properties that logarithms possess.

(a) $\log_4 5$

$$\begin{aligned}\log_4 5 &= \log_4 (3^{\log_3 5}) \\ &= \log_4 3^q \\ &= q \log_4 3 \\ &= qp,\end{aligned}$$

which is the correct answer for part (a).

(b) $\log_{10} 5$

$$\begin{aligned}\log_{10} 5 &= \log_{10} (3^{\log_3 5}) \\ &= \log_{10} 3^q \\ &= q \log_{10} 3 \\ &= q \times \frac{1}{\log_3 10} \\ &= \frac{q}{\log_3 5 + \log_3 2} \\ &= \frac{q}{q + \log_3 (4^{1/2})} \\ &= \frac{q}{q + \frac{1}{2} \log_3 4} \\ &= \frac{q}{q + \frac{1}{2} \times \frac{1}{\log_4 3}} \\ &= \frac{q}{q + \frac{1}{2p}} \\ &= \frac{2pq}{2p \left(q + \frac{1}{2p} \right)} \\ &= \frac{2pq}{2pq + 1},\end{aligned}$$

which is the correct answer for part (b).