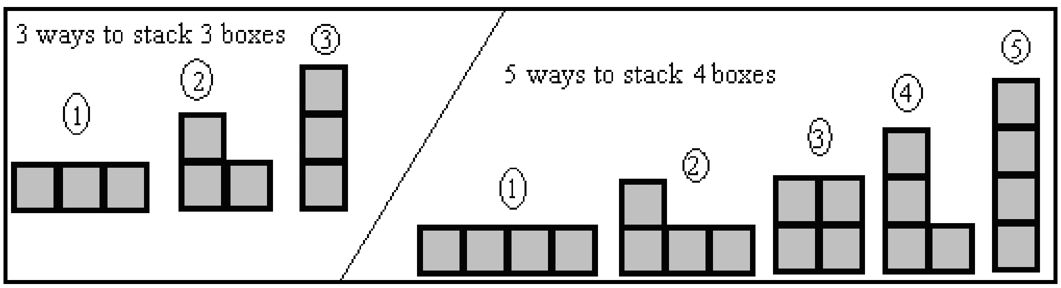
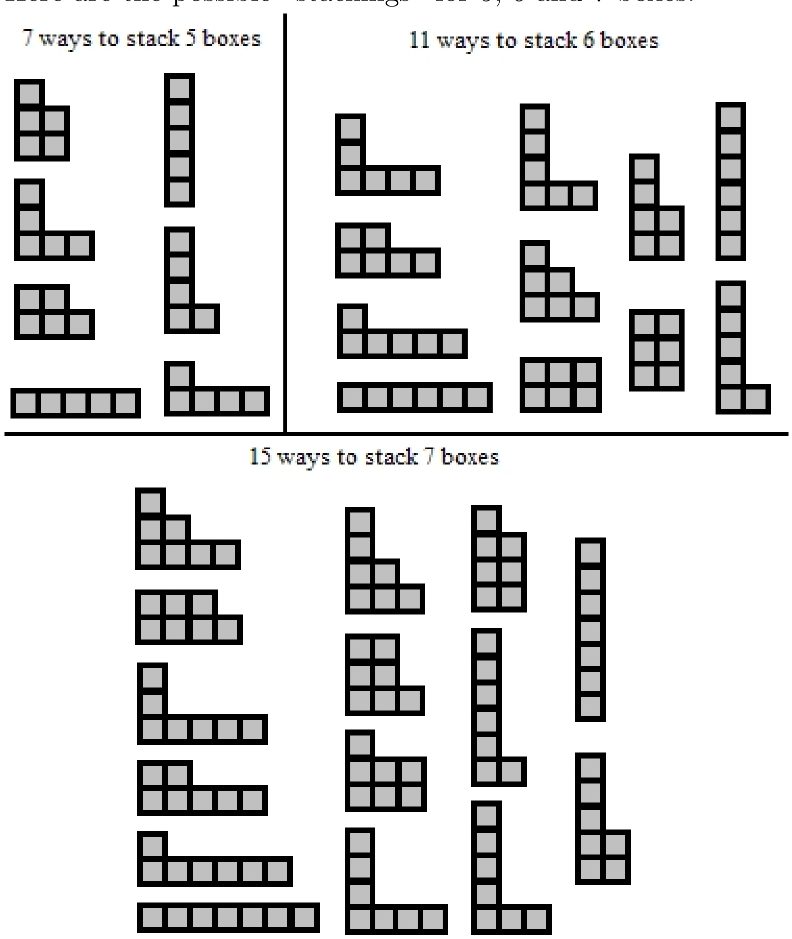
Eastern Shore High School Mathematics Competition
2007 Team Contest Solutions

1. We will stack boxes in such a way that each "layer" (or row) of boxes has at least as many boxes as the layer above it, and on each "layer" all the boxes must be pushed as far as possible to the left. For example, the diagram below shows all of the possible ways to stack three boxes or four boxes under these rules. Note that there are three different ways to stack three boxes, and there are five different ways to stack four boxes.



(a) Draw diagrams (similar to the above diagram) for 5 boxes, 6 boxes and 7 boxes. (Use the provided manipulatives to get a feel for this question; make sure you don't miss any possibilities!)
Here are the possible "stackings" for 5, 6 and 7 boxes:



These diagrams may be used to answer parts (b) and (c):

(b) How many ways are there to stack 5 boxes using an even number of layers? 6 boxes? 7 boxes?
Answer: There are 3, 6, and 7 ways, respectively, to stack the boxes in such a way.

(c) How many ways are there to stack 5 boxes in such a way that the bottom layer has an even number of boxes? 6 boxes? 7 boxes?

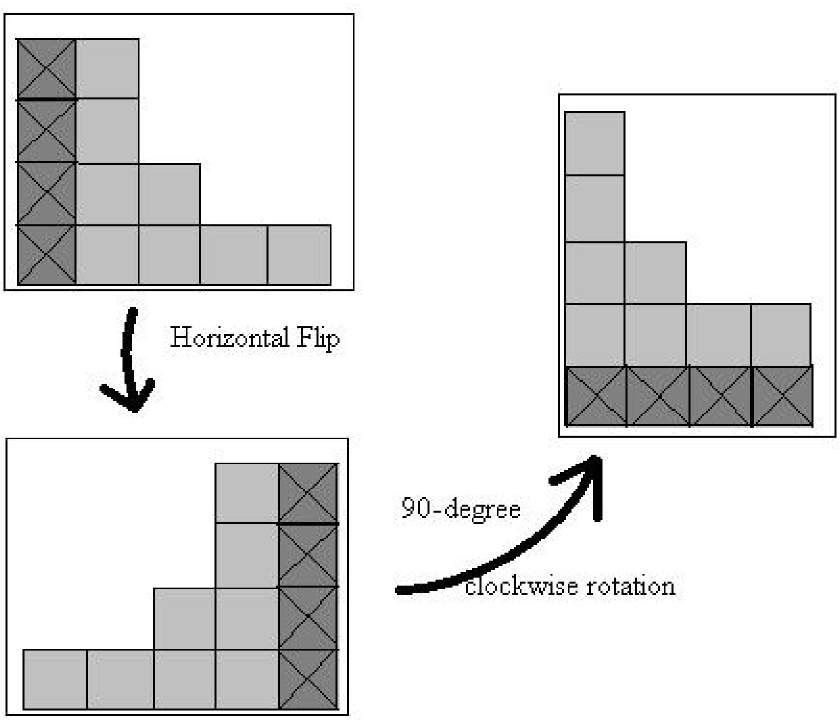
Answer: There are 3, 6 and 7 ways, respectively, to stack the boxes in such a way.

(d) Based on your answers to (b) and (c), you should be able to conjecture a general connection between the number of ways to stack N boxes using an even number of layers and the number of ways to stack N boxes in such a way that the bottom layer has an even number of boxes. State this conjecture in words, and explain why it will be true for all values of N.

The connection is that the number of ways to stack N boxes using an even number of layers is equal to the number of ways to stack N boxes such that the bottom later has an even number of boxes.

This is indeed true for all N. This can be seen by manipulating the diagrams. Consider any stacking with an even number of layers. Flip this diagram horizontally, and then rotate it 90 degrees clockwise. The result is a stacking whose bottom later has an even number of boxes. This connection works in reverse as well - in fact, through this manipulation of diagrams, we see that there is a one-to-one correspondence between diagrams of the first type (even number of layers) and the second type (bottom layer is of even length). Since this will work regardless of the number of boxes in the diagram, this one-to-one correspondence will hold for all N.

Here's an example using $N = 12$ boxes:



Notice the darker boxes - the leftmost box of each layer in the original diagram becomes a box in the bottom layer of the new diagram.

2. $\triangle\, ABC$ and $\triangle\, CDA$ are equilateral triangles (sharing side $\overline{AC}$), point E is the midpoint of $\overline{AB}$, and $\overleftrightarrow{DE}$ intersects $\overline{AC}$ at point F. The length of side $\overline{CD}$ is $\sqrt{7}$. Find the EXACT length (not just a decimal approximation) of $\overline{AF}$.

   Solution # 1:

   We will draw a diagram of the described figures (shown on the following page); for convenience, we will locate point A at the origin of a coordinate plane. Since $\triangle\, ABC$ and $\triangle\, CDA$ both have side length $\sqrt{7}$, we know that the x-coordinate of B must be $\frac{\sqrt{7}}{2}$, and its y-coordinate must be $\sqrt{3}\cdot\frac{\sqrt{7}}{2} = \frac{\sqrt{21}}{2}$. Similarly, the coordinates of D must be $\left(\frac{\sqrt{7}}{2}, -\frac{\sqrt{21}}{2}\right)$. E is the midpoint of $A(0,0)$ and $B\left(\frac{\sqrt{7}}{2}, \frac{\sqrt{21}}{2}\right)$, so we have $E\left(\frac{\sqrt{7}}{4}, \frac{\sqrt{21}}{4}\right)$.

   Now that we have the coordinates of D and E, we can find the equation of $\overleftrightarrow{DE}$. This line has slope

$$\begin{aligned} m_{\overleftrightarrow{DE}} &= \frac{\frac{\sqrt{21}}{4} - (-\frac{\sqrt{21}}{2})}{\frac{\sqrt{7}}{4} - \frac{\sqrt{7}}{2}} \\ &= \frac{3\sqrt{21}/4}{-\sqrt{7}/4} \\ &= -\frac{3\sqrt{21}}{\sqrt{7}} \\ &= -3\sqrt{3} \end{aligned}$$

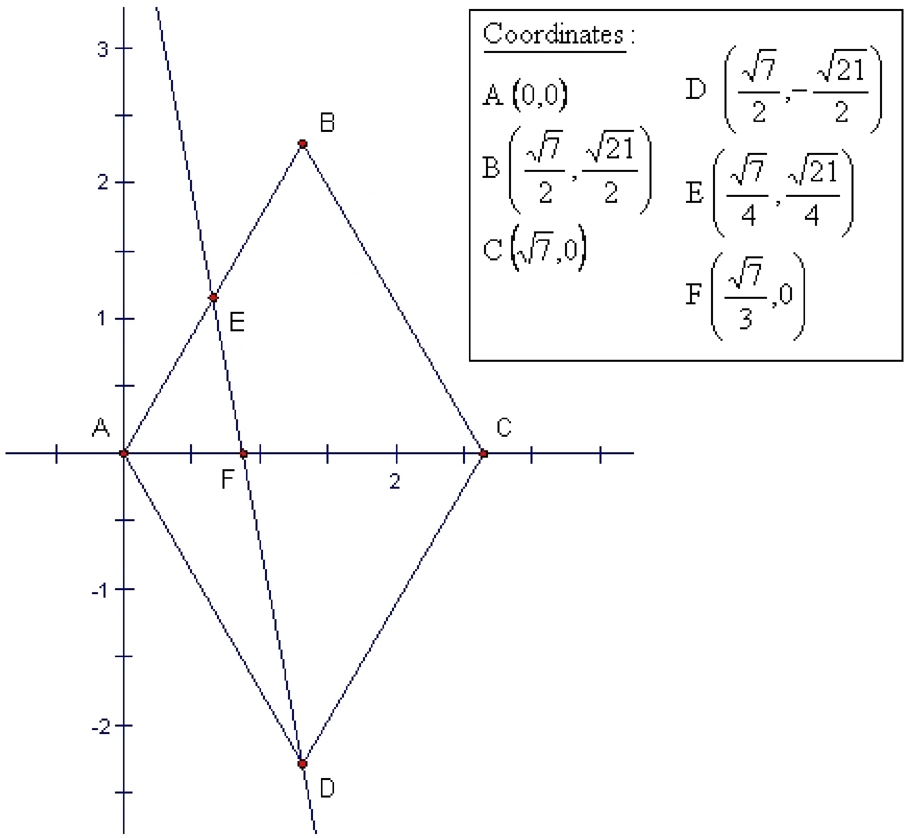
   Using the point-slope equation for a line (with the coordinates of E), we obtain the equation

$$y - \frac{\sqrt{21}}{4} = -3\sqrt{3}\left(x - \frac{\sqrt{7}}{4}\right).$$

   Since we constructed our diagram in such a way that $\overline{AC}$ lies on the x-axis, point F is simply the intersection of this line with the x-axis; thus, we need only substitute $y = 0$ and solve for x:

$$\begin{aligned} -\frac{\sqrt{21}}{4} &= -3\sqrt{3}\left(x - \frac{\sqrt{7}}{4}\right) \\ -\frac{\sqrt{21}}{4} &= -3x\sqrt{3} + \frac{3\sqrt{21}}{4} \\ -\sqrt{21} &= -3x\sqrt{3} \\ -\sqrt{21} &= -3x\sqrt{3} \\ \frac{\sqrt{21}}{3\sqrt{3}} &= x \\ \frac{\sqrt{7}}{3} &= x \end{aligned}$$

   Thus, the length of $\overline{AF}$ is $\frac{\sqrt{7}}{3}$.



Solution # 2. Sketch the above diagram *without* the coordinate plane; that is, just draw a simple geometric sketch based on what's given. Note the following:

- $\angle EFA$ and $\angle DFC$ are vertical angles; therefore, $m\angle EFA = m\angle DFC$.
- $\triangle\ ABC$ and $\triangle\ ADC$ are equilateral triangles; therefore, $m\angle EAF = m\angle DCF$ (since both measure $60°$).
- Since $\triangle\ EAF$ and $\triangle\ DCF$ have two pairs of congruent angles between them, it follows that $\triangle\ EAF$ is geometrically similar to $\triangle\ DCF$.

Since these two triangles are similar, their corresponding sides' length are proportional. Since E is the midpoint of $\overline{AB}$, $2AE = AB$. Since $AB = CD$, it follows that $2AE = CD$, or (to establish the ratio between pairs of corresponding sides of the two similar triangles), $\dfrac{CD}{AE} = 2$. Looking at another pair of corresponding sides, now, we see that $\dfrac{CF}{AF} = 2$, so $CF = 2AF$. Also, it's clear from the diagram that $AF + CF = AC$. Therefore, by substitution, we have $AF + (2AF) = AC$, which simplifies to $3AF = AC$. Therefore, $AF = \dfrac{AC}{3} = \dfrac{\sqrt{7}}{3}$.

3. A natural number, W, is called a "wonderful number" if each of the following is true:

   - W is a multiple of 8
   - W has a remainder of 2 when divided by 3
   - W has a remainder of 2 when divided by 5

   (a) What is the smallest wonderful number?

Answer: 32. (Easily found by trial-and-error.)

(b) Find all of the wonderful numbers between 0 and 500.

(c) Find a formula for all wonderful numbers.

Solution: It's easiest here to first solve (c), and then use this general result to solve (b). The criteria for a wonderful number indicate that the difference between any two wonderful numbers must be a multiple of 8, a multiple of 3 and a multiple of 5. Therefore, the difference between any two wonderful numbers must be a multiple of $8 \cdot 3 \cdot 5 = 120$. Since we know 32 is a wonderful number, the other wonderful numbers may be found by adding multiples of 120 - thus, a number is a wonderful number if (and only if) it is of the form $32 + 120n$, where n is an integer and $n \geq 0$. It follows that the wonderful numbers less than 500 are 32, 152, 272 and 392.

4. Mike found a great investment opportunity that will increase in value by 50% each week.

Suppose he invests \$10 today...

Give whole number answers for questions (a), (b) and (c).

(a) After how many weeks will Mike have over one million dollars?

(b) After how many weeks will Mike have over one billion dollars?

(c) After how many weeks will Mike have over one trillion dollars?

(d) After how many weeks will Mike have x dollars, where $x > 10$?

Solution: We will solve (d) first, and use the result to find answers for (a), (b) and (c).

Mike's investment increases in value by a factor of 1.5 per week, every week; therefore, the value in dollars (x) of his investment after n weeks is $x = 10(1.5)^n$. Solve for n:

$$\begin{aligned} x &= 10(1.5)^n \\ \frac{x}{10} &= (1.5)^n \\ \log_{10}\left(\frac{x}{10}\right) &= n \log_{10} 1.5 \\ n &= \frac{\log_{10}\left(\frac{x}{10}\right)}{\log_{10} 1.5}, \end{aligned}$$

or (equivalently) $n = \dfrac{\log_{10} x - 1}{\log_{10} 1.5}$.

Now that we have the general solution for x, we simply solve for n when $x = 10^6, 10^9$, and 10^{12} to find answers for (a), (b) and (c), respectively. (In each case, the answer should be rounded *up* to the next whole number.)

(a)

$$
\begin{aligned}
n &= \frac{\log_{10}(10^6) - 1}{\log_{10} 1.5} \\
&= \frac{6 - 1}{\log_{10} 1.5} \\
&= \frac{5}{\log_{10} 1.5} \\
&\approx 28.4,
\end{aligned}
$$

so the answer is 29 weeks.

(b)

$$
\begin{aligned}
n &= \frac{\log_{10}(10^9) - 1}{\log_{10} 1.5} \\
&= \frac{9 - 1}{\log_{10} 1.5} \\
&= \frac{8}{\log_{10} 1.5} \\
&\approx 45.4,
\end{aligned}
$$

so the answer is 46 weeks.

(c)

$$
\begin{aligned}
n &= \frac{\log_{10}(10^{12}) - 1}{\log_{10} 1.5} \\
&= \frac{12 - 1}{\log_{10} 1.5} \\
&= \frac{11}{\log_{10} 1.5} \\
&\approx 62.5,
\end{aligned}
$$

so the answer is 63 weeks.