

Eastern Shore High School Mathematics Competition  
2007 Individual Contest Solutions

Choose the best answer to each of the following 20 questions. Your score will be the number of correct answers you give out of 20 possible. There is no penalty for incorrect answers. In the event of tie scores, questions #18, #19 and #20 (in that order) will be used as tiebreakers.

1. If  $f(g(x)) = x^4$  and  $f(x) = (x + 1)^2$ , then  $g(x)$  is which of the following?

- (a)  $x^2 - 1$     (b)  $x^2 + 1$     (c)  $x^4 + 1$     (d)  $(x + 1)^4$     (e)  $\frac{x^4}{(x + 1)^2}$

Solution: (a) is the correct solution:

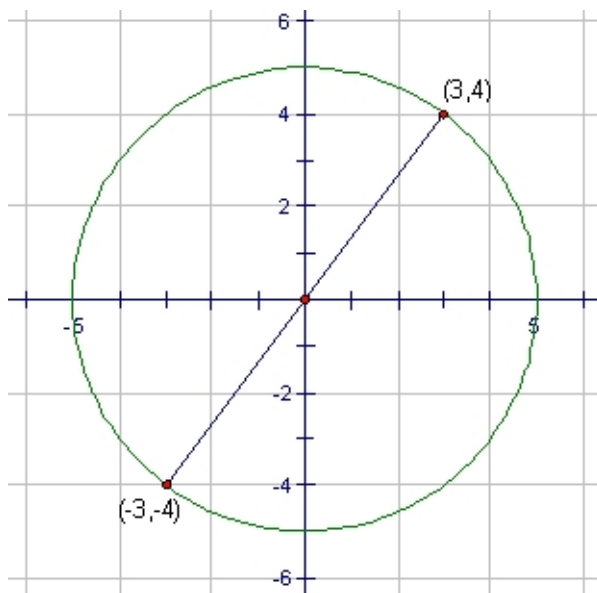
$$f(g(x)) = f(x^2 - 1) = ((x^2 - 1) + 1)^2 = (x^2)^2 = x^4.$$

2. A chord of length 10 of the circle  $x^2 + y^2 = 25$  has one of its endpoints at (3,4). What are the coordinates of the chord's other endpoint?

- (a) (-4,3)    (b) (-3,-4)    (c) (-3,4)    (d) (3,-4)    (e) (4,3)

Solution: (b) (-3,-4)

The circle  $(x - 0)^2 + (y - 0)^2 = 5^2$  is centered at the origin and has radius 5. Therefore, a "chord of length 10" is actually a *diameter* of the circle. If a circle's center is at (0,0) and one end of a diameter is at (3,4), the other end of the diameter will be at (-3,-4):



3. Four pieces of luggage were checked at an airline counter. Their weights (in pounds) are 32, 28, 30 and 37. Now, a fifth piece of luggage is checked. The mean and the median weight of the given pieces of luggage are now the same. Which of the following was the weight (in pounds) of the fifth piece of luggage?

- (a) 28    (b) 30    (c) 31    (d) 33    (e) 35

Solution: (d) 33

If we add 33 to the existing sample, then the mean and the median will both be 32.

4. Starting with the number 8: first, increase the number by 10%. Next, decrease the resulting number by 10%, then increase that number by 20%, and finally, decrease that number by 20%, finishing with a number, which we'll call  $x$ . Which of the following is true?

- (a)  $7.25 \leq x < 7.5$
- (b)  $7.5 \leq x < 7.75$
- (c)  $7.75 \leq x < 8$
- (d)  $8 \leq x < 8.25$
- (e)  $8.25 \leq x < 8.5$

Solution: (b)  $7.5 \leq x < 7.75$ . (Exact answer:  $x = 7.6032$ )

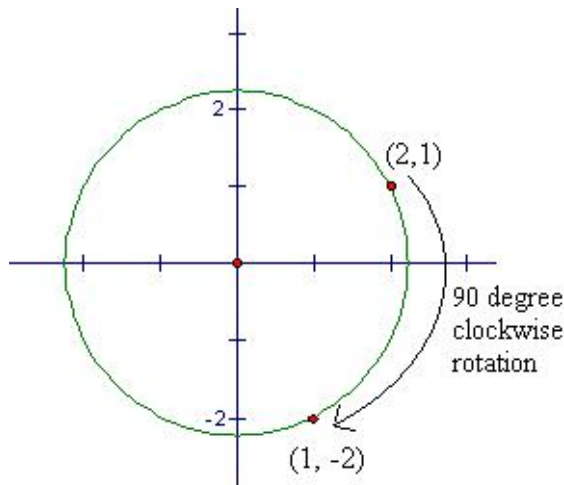
After each step, we have:

- Increase by 10%:  $8 + (0.1)8 = 8.8$
- Decrease by 10%:  $8.8 - (0.1)8.8 = 8.8 - 0.88 = 7.92$
- Increase by 20%:  $7.92 + (0.2)7.92 = 7.92 + 1.584 = 9.504$
- Decrease by 20%:  $9.504 - (0.2)9.504 = 9.504 - 1.9008 = 7.6032$

5. If the point  $(2,1)$  is rotated about the origin by 90 degrees in a clockwise direction, the coordinates of its image are which of the following?

- (a)  $(-1,2)$     (b)  $(1,-2)$     (c)  $(-2,1)$     (d)  $(2,-1)$     (e) None of these

Solution: (b)  $(1, -2)$



6. If  $f(x) = 3^{x/3}$ , then what is the value of  $f\left(\log_6 \frac{1}{216}\right)$ ?

- (a) -27    (b) -3    (c)  $\frac{1}{216}$     (d)  $\frac{1}{27}$     (e)  $\frac{1}{3}$

Solution: (e)  $\frac{1}{3}$

$$f\left(\log_6 \frac{1}{216}\right) = f\left(\log_6 \frac{1}{6^3}\right) = f(\log_6 6^{-3}) = f(-3) = 3^{-3/3} = 3^{-1} = \frac{1}{3}.$$

7. If  $f(x) = \frac{1}{x^3}$ , then  $f(f(f(f(x))))$  is which of the following?

- (a)  $x^3 - 1$     (b)  $1/x^9$     (c)  $x^{81}$     (d)  $1/x^{81}$     (e) None of these

Solution: (c)  $x^{81}$

We will rewrite  $f(x) = \frac{1}{x^3}$  in the form  $f(x) = x^{-3}$  and repeatedly apply the power-of-a-power rule:

$$\begin{aligned} f(f(f(f(x)))) &= \left( \left( \underbrace{(x^{-3})^{-3}}_{x^9} \right)^{-3} \right)^{-3} \\ &= \left( \underbrace{(x^9)^{-3}}_{x^{-27}} \right)^{-3} \\ &= (x^{-27})^{-3} \\ &= x^{81} \end{aligned}$$

8. Assuming  $\pi \approx 3$ , consider a sphere of radius 5 inscribed in a cube. What is the approximate volume of the region contained *between* the sphere and the cube?

- (a) 300    (b) 400    (c) 500    (d) 600    (e) 700

Solution: (c) 500

A cube of radius 5 can be inscribed in a cube of side length 10, so the cube's volume is 1000. The sphere's volume, using the estimate  $\pi \approx 3$ , is given by

$$V = \frac{4}{3}\pi r^3 \approx \frac{4}{3}(3)(5^3) = 4 \cdot 125 = 500.$$

Therefore, the approximate volume of the region inside the cube but outside the sphere is  $1000 - 500 = 500$ .

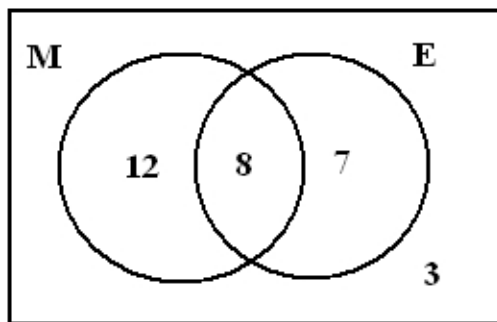
9. Of thirty students at East Coast High School, 20 are taking math, 15 are taking English, and 8 are taking both. If one of these 30 students is randomly selected, what is the probability that the student is taking neither math nor English?

- (a) 3/30    (b) 8/30    (c) 22/30    (d) 27/30    (e) None of these

Solution: (a) 3/30

The number of students taking math or English (or both) is  $20+15-8=27$ . Since there are 30 students at the school,  $30-27=3$  take neither subject. Therefore, if we choose at random from among 30 students, there is a 3 in 30 chance, or a probability of 3/30, that we'll select a student who is taking neither math nor English.

This can also be illustrated via Venn diagram; let M and E denote the sets of students (out of the original 30) taking Math and English, respectively:



$20 - 8 = 12$  students in M but not in E

$15 - 8 = 7$  students in E but not in M

Thus,  $12 + 8 + 7 = 27$  students take math or English (or both),  
leaving  $30 - 27 = 3$  students who take neither subject.

10. If  $(2^4)(2^{4n})(4) = 16$ , then what is the value of  $n$ ?

- (a) -2    (b) -1/2    (c) 0    (d) 4    (e) None of these

Solution: (b) -1/2

Rewrite both sides as powers of 2, then equate the exponents and solve for  $n$ :

$$\begin{aligned} (2^4)(2^{4n})(4) &= 16 \\ (2^4)(2^{4n})(2^2) &= 2^4 \\ 2^{4+4n+2} &= 2^4 \\ 4 + 4n + 2 &= 4 \\ 4n + 2 &= 0 \\ n &= -1/2 \end{aligned}$$

11. The number 2007 has how many distinct prime factors?

- (a) 1    (b) 2    (c) 3    (d) 4    (e) more than 4

Solution: (b) 2

$2007 = 3^2 \cdot 223$ , so the *distinct* prime factors are 3 and 223.

12. A piece of paper is 0.01 inches thick. It is folded in such a way that its thickness is doubled by each fold. If the paper is folded 5 times, how thick is the result (in inches)?

- (a) 0.02    (b) 0.05    (c) 0.10    (d) 0.31    (e) 0.32

Solution: (e) 0.32

Each fold doubles the width, so we multiply by 2 five times to find the final width:

$$\text{Width} = 0.01 \cdot (2^5) = 0.32 \text{ inches.}$$

13. Find the solution set of the following equation.

$$10^{\log_{10}(x+1) + 2\log_{10}(x)} - 3^{\log_3(5)} = x^3 + 4x$$

- (a)  $\{-1, 5\}$     (b)  $\{0\}$     (c)  $\{5\}$     (d)  $\{5, 10\}$     (e)  $\emptyset$

Solution: (c)  $\{5\}$

We can rewrite the equation as follows:

$$10^{\log_{10}(x+1)} \times 10^{2\log_{10}(x)} - 3^{\log_3(5)} = x^3 + 4x$$

$$10^{\log_{10}(x+1)} \times \left(10^{\log_{10}(x)}\right)^2 - 3^{\log_3(5)} = x^3 + 4x$$

At this point, apply the identity  $b^{\log_b(a)} = a$  throughout, and then solve for  $x$ :

$$(x+1)(x)^2 - 5 = x^3 + 4x$$

$$x^3 + x^2 - 5 = x^3 + 4x$$

$$x^2 - 4x - 5 = 0$$

$$(x+1)(x-5) = 0$$

$$x = -1 \quad , \quad x = 5$$

This appears to indicate the solution set is  $\{-1, 5\}$ . However, if both potential solutions are tested in the original equation, we see that  $x = -1$  is *not* a valid solution, since substituting  $x = -1$  would require us to evaluate  $\log_{10}(0)$  and  $\log_{10}(-1)$ , neither of which exists. Therefore, the only actual solution to the original equation is  $x = 5$ .

14. Suppose the number  $48m25n$  (where  $m$  and  $n$  are unknown digits) is divisible by 9. Evaluate  $m + n$ .

- (a) 2    (b) 8    (c) 9    (d) 18    (e) Cannot be determined from the information given.

Solution: (e) Cannot be determined from the information given.

The number is divisible by 9 if and only if the sum of its digits is a multiple of 9. The sum of the given number's digits is  $19 + (m + n)$ , which is a multiple of 9 if  $m + n = 8$  or if  $m + n = 17$ .

NOTE: Even though  $m + n = 8$  is a solution, (b) cannot be considered the correct answer to this question, since the value of  $m + n$  cannot be uniquely determined from the information given. As stated in the exam instructions, students are to choose the *best* answer for each question. In this case, (b) *might* be correct, but (e) is *definitely* correct, so (e) is unambiguously the best answer to this question.

15. Charlie and Mac sit down to eat some brown and red M&M's. Initially, Charlie has twice as many brown M&M's as red, while Mac has 6 more red M&M's than twice his number of brown. Mac then gives Charlie 5 red M&M's, after which Charlie has 20 more brown than red, while Mac has 18 more red than brown. How many total M&M's do Charlie and Mac have?

- (a) 67    (b) 75    (c) 127    (d) 132    (e) 137

Solution: (d) 132

Let  $r_1, b_1$  denote Charlie's initial number of red and brown M&M's, respectively. Similarly, let  $r_2, b_2$  denote Mac's initial number of red and brown M&M's, respectively. Then, the second sentence of the problem statement tells us the following:

$$(1) \quad b_1 = 2r_1$$

$$(2) \quad 2b_2 + 6 = r_2$$

After Mac gives Charlie 5 red M&M's, Charlie has  $r_1 + 5$  red M&M's and Mac has  $r_2 - 5$  red M&M's. Therefore, the third sentence of the problem statement tells us:

$$(3) \quad b_1 = (r_1 + 5) + 20$$

$$(4) \quad b_2 + 18 = (r_2 - 5)$$

Substitute  $b_1 = 2r_1$  into equation (3) above to solve for  $r_1$  and  $b_1$ :

$$2r_1 = r_1 + 5 + 20 \longrightarrow r_1 = 25, \text{ which implies } b_1 = 2 \cdot 25 = 50.$$

Next, substitute  $r_2 = 2b_2 + 6$  into equation (4) above to solve for  $r_2$  and  $b_2$ :

$$b_2 + 18 = (2b_2 + 6) - 5 \longrightarrow b_2 = 17, \text{ which implies } r_2 = 2 \cdot 17 + 6 = 40.$$

So, Charlie starts with 25 red M&M's and 50 brown M&M's, for a total of 75 M&M's, while Mac starts with 40 red M&M's and 17 brown M&M's, for a total of 57 M&M's. Thus, there are  $75 + 57 = 132$  M&M's altogether.

16. Everyone at the party is wearing a blue shirt. Everyone on the lacrosse team is at the party. Everyone at the party who owns a dog is on the lacrosse team. Jason is wearing a blue shirt, and he owns a dog. Of the following statements, which one(s) *must* be true?

- I. Jason is at the party.
- II. Jason is on the lacrosse team.
- III. If Jason is at the party, then he is on the lacrosse team.

- (a) II and III must be true, but not I.
- (b) I, II and III all must be true.
- (c) Only I must be true.
- (d) Only II must be true.
- (e) Only III must be true.

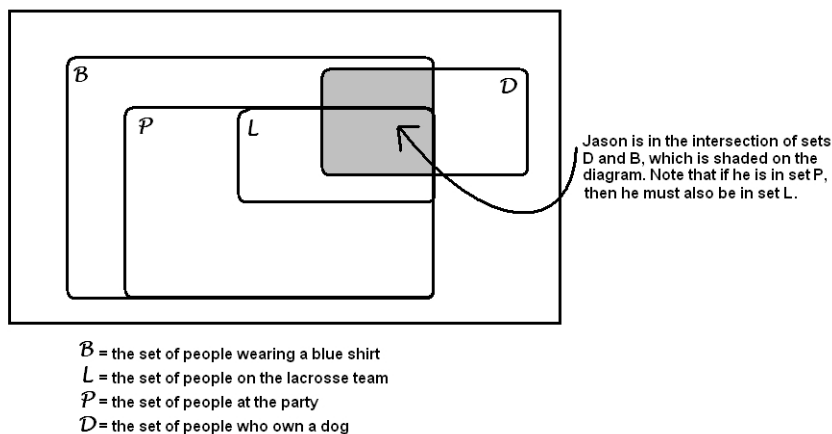
Solution: (e) Only III must be true.

First, notice that none of the original premises state that anything *follows from* the fact that Jason is wearing a blue shirt. We are given that everyone at the party is wearing a blue shirt; however, the converse of this statement is *not* necessarily true – that is, a person wearing a blue shirt may or may not be at the party. Therefore, the fact that Jason is wearing a blue shirt does not require him to be at the party.

Next, we note that nothing *necessarily* follows from the fact that Jason owns a dog. The only premise involving dog owners states that everyone *at the party* who owns a dog is on the lacrosse team. As noted earlier, Jason is not necessarily at the party. However, he *does* own a dog, so we can conclude that *if* Jason is at the party, *then* he must be on the lacrosse team.

Therefore, while neither statement I nor II is necessarily true, the conditional statement III must be true.

Note: This situation can be modeled with a Venn diagram, as follows:



17. A positive integer  $N$  has a prime factorization given by the product of five distinct primes. How many different factors does  $N$  have?

- (a)  $2^5$       (b) 5      (c)  $5^2$       (d)  $5 \times 5 \times 5 \times 5 \times 5$   
 (e) It cannot be determined because the primes are unknown.

Solution: (a)  $2^5$

If a number is the product of 5 distinct primes, then there are  $2^5 = 32$  ways to select a subset of those primes, and each subset's product gives us one factor of the number. More generally, if a number is the product of  $n$  distinct primes, then that number has  $2^n$  factors.

For example:  $30 = 2 \cdot 3 \cdot 5$ , which is the product of 3 distinct primes, so it has  $2^3 = 8$  factors:

$$\begin{array}{ccccccc} 1 \cdot 1 \cdot 1 = 1 & 2 \cdot 1 \cdot 1 = 2 & 1 \cdot 3 \cdot 1 = 3 & 1 \cdot 1 \cdot 5 = 5 & & & \\ 2 \cdot 3 \cdot 1 = 6 & 2 \cdot 1 \cdot 5 = 10 & 1 \cdot 3 \cdot 5 = 15 & 2 \cdot 3 \cdot 5 = 30 & & & \end{array}$$

Note: the smallest positive integer with five distinct prime factors (and therefore the smallest positive integer with exactly 32 factors) is  $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$ .

18. (First tiebreaker) At bedtime, Homer cuts a pie into three equal pieces. When Homer isn't looking, Bart steals two of the pieces, leaving only one piece for Homer.

Grumbling, Homer immediately eats one third of the remaining piece of pie.

Exactly one hour later, Homer eats one third of what's left (of his piece).

Exactly half an hour after that, Homer eats one third of what's left.

Exactly a quarter of an hour after that, Homer eats one third of what's left.

Exactly an eighth of an hour after that, Homer eats one third of what's left...

...and so on. If Homer continues eating in this way, what fraction of the original pie will Homer have eaten by the time Marge gets up the next morning?

- (a) one ninth      (b) one sixth      (c) one fourth      (d) one third      (e) one half

Solution: (d) one third - he would finish his one piece of the original pie.

Homer starts with one third of the pie, and after each round of eating, he's left with two-thirds of what he had previously. Therefore, after one hour, Homer is left with  $\frac{2}{3}$  of his  $\frac{1}{3}$ , or  $\frac{2}{9}$ , of the original pie. Half an hour later, he has  $\frac{2}{3} \cdot \frac{2}{9} = \frac{2}{27}$  of the pie remaining, then  $\frac{2}{3} \cdot \frac{2}{27} = \frac{2}{81}$  of the pie remaining, and so on. In general, after "Homer eats one third of what's left"  $n$  times, he is left with  $\frac{2}{3} \cdot \left(\frac{2}{3}\right)^n$  of the original pie. As  $n$  increases without bound, this quantity converges to 0; therefore, "if Homer continues eating in this way," he will ultimately be left with nothing - thus, he'll have finished his original piece, which was one third of the original pie.

Next, consider the frequency with which Homer eats:

- First time: after 1 hour
- Second time: after  $1 + \frac{1}{2} = \frac{3}{2}$  hours
- Third time: after  $1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$  hours
- Fourth time: after  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$  hours

...and so on. Homer will have eaten "one-third of what's left"  $n$  times after  $1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}} = \frac{2^n - 1}{2^{n-1}}$  hours,

which is always less than two hours. Therefore, if the eating process continues as described, then Homer will *finish* his piece of pie in two hours - which will certainly be before “the time Marge gets up the next morning.”

19. (Second tiebreaker) A “perfect number” is a positive integer,  $n$ , that is equal to the sum of all of the factors of  $n$  that are less than  $n$ . For example, the factors of 6 that are less than 6 are 1, 2 and 3; since  $6=1+2+3$ , 6 is a perfect number.

Which of the following statements, if any, are true?

- I. 28 is a perfect number.
- II. 17 is not a perfect number.
- III. 496 is a perfect number.

- (a) Only I is true.
- (b) I and II are true and III is false.
- (c) I and III are true and II is false.
- (d) I, II and III are all true.
- (e) None of the above combinations is correct.

Solution: (d) All three statements are true.

- The factors of 28 that are less than 28 are 1,2,4,7, and 14, which sum to 28. Therefore, 28 is perfect.
- The factor of 17 that is less than 17 is 1, which sums to 1. Therefore, 17 is not perfect.
- The factors of 496 that are less than 496 are 1,2,4,8,16,31,62,124, and 248, which sum to 496.

Note: An even number,  $x$ , is perfect if and only if it can be written in the form  $x = 2^n(2^{n+1} - 1)$  and  $2^{n+1} - 1$  is prime. For example:

- $6 = 2^1(2^2 - 1)$ , and  $2^2 - 1 = 3$  which is prime, so 6 is perfect.
- $28 = 2^2(2^3 - 1)$ , and  $2^3 - 1 = 7$  which is prime, so 28 is perfect.
- $496 = 2^4(2^5 - 1)$ , and  $2^5 - 1 = 31$  which is prime, so 496 is perfect.
- On the other hand:  $120 = 2^3(2^4 - 1)$ , but  $2^4 - 1 = 15$  which is not prime, so 120 is not perfect.

20. (Third tiebreaker) Suppose an integer sequence of numbers  $a_n, n = 1, 2, 3, \dots$  is defined as follows:

$$a_1 = 1, a_2 = 2, \text{ and } a_{n+1} = 2a_n + a_{n-1} \text{ for } n \geq 2.$$

Assuming the ratio of consecutive terms,  $\frac{a_{n+1}}{a_n}$ , approaches a fixed number  $L$  as  $n$  gets larger without bound, which of the following is true?

- (a)  $L = \sqrt{2}$
- (b)  $L$  is the Golden Ratio
- (c)  $L = 2.5$
- (d)  $L - 2$  is negative
- (e)  $L = \frac{(L-1)(L+1)}{2}$

Solution: (e)  $L = \frac{(L-1)(L+1)}{2}$

Assume the ratio  $\frac{a_{n+1}}{a_n}$  approaches a fixed number  $L$  as  $n$  gets larger without bound. Then, for “large” values of  $n$ , we can say  $a_{n+1} \approx La_n$ , and also  $a_n \approx La_{n-1}$  (since  $n-1$  also becomes “large” when  $n$  becomes “large”). These two statements can be combined:

$$a_{n+1} \approx La_n \approx L(La_{n-1}) = L^2a_{n-1}.$$

Thus, the original two-term recurrence,

$$a_{n+1} = 2a_n + a_{n-1},$$

can be rewritten as

$$L^2a_{n-1} = 2La_{n-1} + a_{n-1}.$$

Now, assuming such a value  $L$  exists, we solve for it:

$$\begin{aligned} L^2a_{n-1} &= 2La_{n-1} + a_{n-1} \\ a_{n-1}(L^2) &= a_{n-1}(2L + 1) \\ L^2 &= 2L + 1 \\ L^2 - 1 &= 2L \\ \frac{L^2 - 1}{2} &= L \\ \frac{(L - 1)(L + 1)}{2} &= L \end{aligned}$$

Notes:

- (a) The exact value of  $L$  turns out to be  $1 + \sqrt{2}$ ; however, as demonstrated above, it's not necessary to actually solve for  $L$  to find the correct answer to this question.
- (b) If we solve the equation  $L^2 = 2L + 1$  for zero, we get on one side of the equation the polynomial  $L^2 - 2L - 1$ ; this polynomial is called the “characteristic polynomial” of the two-term recursion  $a_{n+1} = 2a_n + a_{n-1}$ .