2006 Team Solutions:

1. Geoboard problem.

   Parts (a) and (c): When you construct polygons so that they have 4 boundary pegs, the area of your polygons should be:

   - $A = 2$ when there is one interior peg
   - $A = 3$ when there are two interior pegs
   - $A = 4$ when there are three interior pegs

   Note that each time we add one interior peg, the area increases by one as well – this implies a linear relationship with a slope of 1. This indicates that, in general, the area of a polygon with 4 boundary pegs will be $i + 1$, where i is the number of interior pegs.

   Parts (b) and (d): When you construct polygons so that they have 2 interior pegs, the area of your polygons should be:

   - $A = 3$ when there are four boundary pegs
   - $A = 3.5$ when there are five boundary pegs
   - $A = 4$ when there are six boundary pegs

   Note that each time we add one boundary peg, the area increases by one half – this implies a linear relationship with a slope of 1/2. This indicates that, in general, the area of a polygon with 4 boundary pegs will be $\frac{b}{2} + 1$, where b is the number of boundary pegs.

   In general, the area of a geoboard figure is $i + \frac{b}{2} - 1$, where i is the number of interior pegs and b is the number of boundary pegs.

   (For more information on this type of problem, search for "Pick's Theorem" on the internet.)

2. In order to determine how many different teams must exist, it makes life easier to call the teams something generic like T_1, T_2, etc. (That is, just identify different teams by subscripts, rather than inventing new names for each.)

   So, since Ed has three favorite teams, let's call these teams T_1, T_2 and T_3. Now, Ed and Fran must have exactly one favorite team in common; let's say that team is team T_1. Then, Fran's other two teams must be teams not in Ed's list – say, T_4 and T_5.

   Thus, so far we have:

   - Ed's teams: T_1, T_2, T_3
   - Fran's teams: T_1, T_4, T_5

   Now we come to Gus. To prove that there must be more than five elbowball teams, we must establish that Gus has a favorite team that has not already been included in Ed's or Fran's list of favorites. But with a bit of thought, we see that this is obvious: Gus has one favorite team in common with Ed, and one favorite in common with Fran, accounting for (at most) two of his three favorite teams. Therefore, Gus must have at least one favorite team that is in neither Ed's nor Fran's list. Thus, there are more than five elbowball teams in all.

(Note: This actually shows that the number of elbowball teams must be 6 or 7. There are 7 teams if Gus has team T_1 as a favorite, but there are only 6 teams if Gus has, for example, teams T_2 and T_4 as favorites.)

3. Correct answer: 79 coconuts were gathered.

   Let N be the number of coconuts gathered.

   - The first man divides these into three equal groups, with one left over. He keeps one of these groups and leaves the other two groups in the pile (and gives the remaining one coconut to the monkey).

     Let x_1 denote the size of each of the three groups of coconuts. Then, $N = 3x_1 + 1$. The first man keeps x_1 coconuts, and the remaining pile contains $2x_1$ coconuts.

   - Later, the second man finds the pile containing $2x_1$ coconuts. He divides these into three equal groups, with one left over. He keeps one of these groups and leaves the other two groups in the pile (and gives the remaining one coconut to the monkey).

     Let x_2 denote the size of each of these three equal groups of coconuts. Then, $2x_1 = 3x_2+1$. (Take a moment to make sure you understand this step! It just writes the number of coconuts in the current pile in two different ways - once in terms of x_1, and then again in terms of x_2.) The second man keeps x_2 coconuts, and the remaining pile contains $2x_2$ coconuts.

     Now, let's solve for N in terms of x_2. We know $x_1 = \dfrac{3x_2+1}{2}$, and we have from the preceding step that $N = 3x_1 + 1$. By substitution, then, we have

     $$N = 3\left(\frac{3x_2+1}{2}\right) + 1 = \frac{9x_2+5}{2}.$$

   - Still later, the third man finds the pile containing $2x_2$ coconuts. He divides these into three equal groups, with one left over. He keeps one of these groups and leaves the other two groups in the pile (and gives the remaining one coconut to the monkey).

     Let x_3 denote the size of each of these three equal groups of coconuts. Then, $2x_2 = 3x_3+1$. (Noticing a pattern?) The second man keeps x_3 coconuts, and the remaining pile contains $2x_3$ coconuts.

     Now, let's solve for N in terms of x_3. We know $x_2 = \dfrac{3x_3+1}{2}$, and we have from the preceding step that $N = \dfrac{9x_2+5}{2}$. By substitution, then, we have

     $$N = \frac{9\left(\frac{3x_3+1}{2}\right)+5}{2} = \frac{\left(\frac{27x_3+9}{2}+\frac{10}{2}\right)}{2} = \frac{27x_3+19}{4}.$$

- Finally, in the morning, the three men find the pile with the remaining $2x_3$ coconuts. They divide these coconuts into three equal groups, with one left over for the monkey.

  Let x_4 denote the size of each of these equal groups of coconuts. Then, $2x_3 = 3x_4 + 1$. Each of the three men keep x_4 coconuts, and finally all coconuts are accounted for.

  Now (one more time!), let's solve for N in terms of x_4. We know $x_3 = \dfrac{3x_4+1}{2}$, and we have from the preceding step that $N = \dfrac{27x_3+19}{4}$. By substitution, then, we have

$$N = \frac{27\left(\frac{3x_4+1}{2}\right)+19}{4} = \frac{\left(\frac{81x_4+27}{2}+\frac{38}{2}\right)}{4} = \frac{81x_4+65}{8}.$$

This will allow us to work backwards - our goal is to find "small" whole number values of x_4 for which N will be a whole number. For example, x_4 can't equal 1, since substituting $x_4 = 1$ would give us $N = \dfrac{146}{8}$, which is not a whole number.

By trial-and-error, we find that the smallest whole number value of x_4 for which N is a whole number is $x_4 = 7$ – in this case, $N = \dfrac{632}{8} = 79$. This is less than 100, so it *should* be our answer!

CHECK: Suppose there were 79 coconuts in the original pile.

- The first man divided the 79 coconuts into 3 equal groups of 26 coconuts, with one left over for the monkey. He took 26 coconuts and left the other 52 in the pile.
- The second man divided the 52 coconuts into 3 equal groups of 17 coconuts, with one left over for the monkey. He took 17 coconuts and left the other 34 in the pile.
- The third man divided the 34 coconuts into 3 equal groups of 11 coconuts, with one left over for the monkey. He took 11 coconuts and left the other 22 in the pile.
- In the morning, the men divided the 22 coconuts into 3 equal groups of 7 coconuts, with one left over for the monkey.

So, it works! Finally, we count up the number of coconuts each man received:

- The first man ends up with $x_1 + x_4 = 26 + 7 = 33$ coconuts.
- The second man ends up with $x_2 + x_4 = 17 + 7 = 24$ coconuts.
- The third man ends up with $x_3 + x_4 = 11 + 7 = 18$ coconuts.

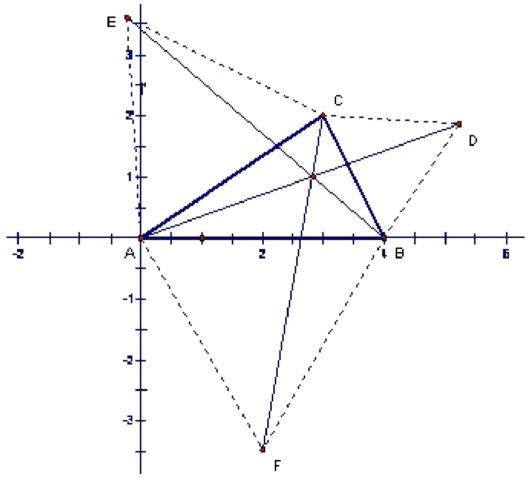
COMMENT: If one knows modular arithmetic, some of these calculations can be simplified (or skipped altogether). For example - we could rewrite the equation relating N to x_4 as follows: $8N - 65 = 81x_4$. This indicates that $8N - 65$ is a multiple of 81, or – in the language of modular arithmetic – $8N \equiv 65 \pmod{81}$. (This is read aloud as "8N is *congruent* to 65 (mod 81)". By properties of modular arithmetic, it is relatively easy to reduce this congruence to: $N \equiv 79 \pmod{81}$. This means that N is a solution to the original problem if (and only if) $N-79$ is a multiple of 81. That means 79, 160, 241, 322, etc. would satisfy the conditions of the problem (aside from the "less than 100 coconuts" condition, which is what results in a unique solution).

4. In the figure shown below, point A is at the origin, point B is at (4,0), and point C is at (3,2). The triangles $\triangle\ ACE$, $\triangle\ CBD$ and $\triangle\ ABF$ are all equilateral.

   (a) Find the sum of the distances from point (2,1) to each of the vertices of $\triangle\ ABC$.

   (b) Lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ all intersect at a single point. Find the coordinates of this point.

   (c) Find the sum of the distances from this intersection point to each of the vertices of $\triangle\ ABC$.



   (a) Using the distance formula, we have:

   - Distance to (0,0): $\sqrt{(2-0)^2+(1-0)^2}=\sqrt{5}$
   - Distance to (3,2): $\sqrt{(2-3)^2+(1-2)^2}=\sqrt{2}$
   - Distance to (4,0): $\sqrt{(2-4)^2+(1-0)^2}=\sqrt{5}$

   So, the sum of these distances is $2\sqrt{5}+\sqrt{2}$ (or approximately 5.886).

   (b) We're given that all three lines meet at a single point. This is true (though by no means obvious), which means that we need only find the intersection point of any *two* of these lines. To do this, we'll need to find the equations of two of the lines. Here, we will find the equations of $\overleftrightarrow{CF}$ and $\overleftrightarrow{AD}$, by finding the coordinates of points F and D and then applying the point-slope equation for a line.

   - Finding F and $\overleftrightarrow{CF}$:
     Since $\triangle\ ABF$ is an equilateral triangle with side length 4, it is not difficult to find the coordinates of F – we need only draw an altitude from F to the midpoint of $\overline{AB}$. This creates a 30-60-90 right triangle with hypotenuse length 4, thus, the altitude's length is $2\sqrt{3}$, and so the coordinates of F are $(2,-2\sqrt{3})$.

It follows that the slope of $\overleftrightarrow{CF}$ is

$$m_{\overleftrightarrow{CF}} = \frac{2-(-2\sqrt{3})}{3-2} = 2+2\sqrt{3}.$$

Therefore, we have the following equation for $\overleftrightarrow{CF}$:

$$\begin{aligned}
y - y_0 &= m_{\overleftrightarrow{CF}}(x - x_0) \\
&\quad \text{(we'll use the coordinates of } C \text{ for } (x_0, y_0)) \\
y - 2 &= (2+2\sqrt{3})(x-3) \\
y &= (2+2\sqrt{3})(x-3)+2, \text{ or} \\
y &= (2+2\sqrt{3})x - (4+6\sqrt{3})
\end{aligned}$$

- Finding D and $\overleftrightarrow{AD}$:
  First, notice that $\overleftrightarrow{AD}$ passes through the point (0,0). Therefore, once we find the coordinates (a, b) of point D, it will immediately follow that the equation of line $\overleftrightarrow{AD}$ is just $y = \frac{b}{a}x$.
  The distance formula gives us $BC = \sqrt{5}$; therefore, $\triangle\ BCD$ is an equilateral triangle with side length $\sqrt{5}$. Therefore, $D(a, b)$ is a point that satisfies both of the following distance equations:

  $$\begin{aligned}
  (a-3)^2 + (b-2)^2 &= 5 \text{ (since } D \text{ is distance } \sqrt{5} \text{ from (3,2), and} \\
  (a-4)^2 + (b-0)^2 &= 5 \text{ (since } D \text{ is distance } \sqrt{5} \text{ from (4,0)}
  \end{aligned}$$

  This is a system of two equations in two variables which can be solved algebraically. (Details omitted – the algebra is straightforward, but takes several lines to complete.) Your result should be:

  $$a = \frac{7}{2} + \sqrt{3}, b = 1 + \frac{\sqrt{3}}{2}.$$

  Therefore, the equation of line $\overleftrightarrow{AD}$ is

  $$y = \frac{1+\frac{\sqrt{3}}{2}}{\frac{7}{2}+\sqrt{3}}x, \text{ or } y = \frac{2+\sqrt{3}}{7+2\sqrt{3}}x.$$

  This can be simplified somewhat by rationalizing the denominator; we may multiply the numerator and denominator each by $7-2\sqrt{3}$ to get the equation

  $$y = \left(\frac{8+3\sqrt{3}}{37}\right)x$$

  for $\overleftrightarrow{AD}$.

Now, we can find the intersection point of the lines by solving the system of equations:

$$\begin{aligned} y &= (2+2\sqrt{3})x - (4+6\sqrt{3}) \\ y &= \frac{8+3\sqrt{3}}{37}x \end{aligned}$$

After quite a bit of algebra, we find that the solution to this system is

$$x = \frac{148+222\sqrt{3}}{66+71\sqrt{3}} = \frac{1014-112\sqrt{3}}{291}, y = \frac{86+60\sqrt{3}}{66+71\sqrt{3}} = \frac{192+58\sqrt{3}}{291}.$$

Or, as decimal approximations, $x \approx 2.818$ and $y \approx 1.005$.

(c) We will now find the distance from this intersection point to each of the vertices of the triangle.

- Point A: $d_1 \approx \sqrt{(2.818-0)^2 + (1.005-0)^2} \approx 2.99$.
- Point B: $d_2 \approx \sqrt{(2.818-4)^2 + (1.005-0)^2} \approx 1.55$.
- Point C: $d_3 \approx \sqrt{(2.818-3)^2 + (1.005-2)^2} \approx 1.01$.

So, the sum of these distances is approximately 5.55.