Eastern Shore High School Mathematics Competition
2006 Individual Contest Solutions

1. What is the 2006^{th} term of the arithmetic progression $2006, 2003, 2000, 1997, \ldots$?

   (a) -6009 (b) -4009 (c) -4006 (d) -2006 (e) 0

   Correct Answer: **(b) -4009**

   The first term is 2006, and each subsequent term is 3 less than the preceding term. Therefore, in progressing from the first term to the 2006th, we subtract 3 2005 times; the result is 2006 - $3 \cdot 2005$ = **-4009**.

2. In the xy-plane, consider the following functions on the points (x, y):

   $f((x, y))$ = the reflection of the point (x, y) across the y-axis

   $g((x, y))$ = the reflection of the point (x, y) across the x-axis

   $h((x, y))$ = the reflection of the point (x, y) across the line $x = 3$

   Given the point (1,2), what is $f(g(h(g(f((1,2))))))$?

   (a) (-7,-2) (b) (-7,2) (c) (1,2) (d) (-6,2) (e) None of these

   Correct Answer: **(b) (-7,2)**

   $f(g(h(g(f((1,2)))))) = f(g(h(g(-1,2)))) = f(g(h(-1,-2))) = f(g(7,-2)) = f(7,2) =$ **(-7, 2)**.

3. What is the product of two numbers whose sum is 10 and whose reciprocals add up to 5/12?

   (a) 12/5 (b) 25/6 (c) 16 (d) 24 (e) None of these

   Correct Answer: **(d) 24**

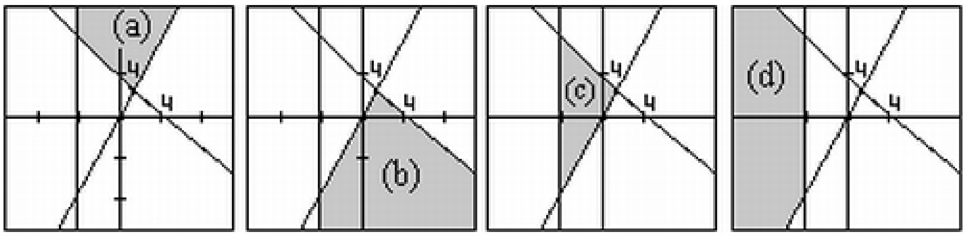
   Let x, y be the two numbers whose sum is 10 and whose reciprocals add up to 5/12. That is, $(x + y) = 10$ and $\frac{1}{x} + \frac{1}{y} = \frac{5}{12}$. We can multiply both sides of the second equation by the common denominator $12xy$ in order to obtain the equivalent equation $12y + 12x = 5xy$. Notice that the left-hand side of this equation can be factored: $12(x + y) = 5xy$. But we also know $x + y = 10$, so, by substitution, we have:

$$\begin{aligned} 12(x+y) &= 5xy \\ 12(10) &= 5xy \\ \frac{120}{5} &= \frac{5xy}{5} \\ 24 &= xy \end{aligned}$$

   Therefore, the product of the two numbers x and y is 24.

   (Notice that this problem can be solved without actually evaluating x and y!)

4. Which region of the plane satisfies $y > 2x$, $y + x < 4$ and $x + 4 > 0$?



(e) None of these

Correct Answer: (c)

The first inequality requires that the shaded region be "above"' the line $y = 2x$. This immediately rules out choice (b). The second inequality, equivalent to $y < 4 - x$, requires that the shaded region be "below" the line $y = 4 - x$. This rules out choice (a). Finally, the third inequality, equivalent to $x > -4$, requires that the shaded region be to the right of the line $x = -4$. This rules out choice (d). Thus, the only graph whose shaded region satisfies all three inequalities is (c).

5. Suppose that Norm is 18 years older than Woody, and that, 13 years ago, Norm was three times as old as Woody. How old is Woody now?

   (a) 5 (b) 9 (c) 22 (d) 40 (e) None of these

   Correct Answer: **(c) 22**.

   Let N=Norm's current age and W=Woody's current age. Then, we are given that $N = W + 18$ and $(N - 13) = 3(W - 13)$. Substituting $W + 18$ for N in the second equation gives us $(W + 18) - 13 = 3W - 39$. Solve for W to get $W = 22$.

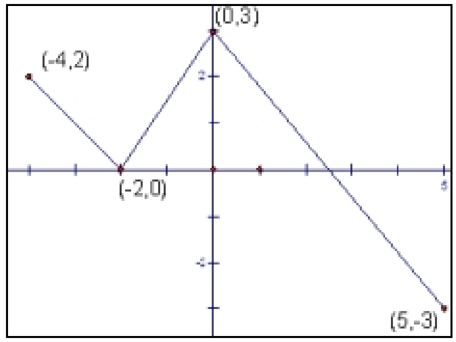
   (Check: If Woody is 22 now, then Norm is 40. Thirteen years ago, they were 9 and 27, respectively. Since 27 is 3 times 9, this answer checks out!)

6. A sample consists of four test scores: 85 , 72 , 91 , 78. Suppose one more test score is added to the sample. Find a value of the fifth test score so that the new sample mean is equal to the new median. (a) 64 (b) 70 (c) 78 (d) 89 (e) 96

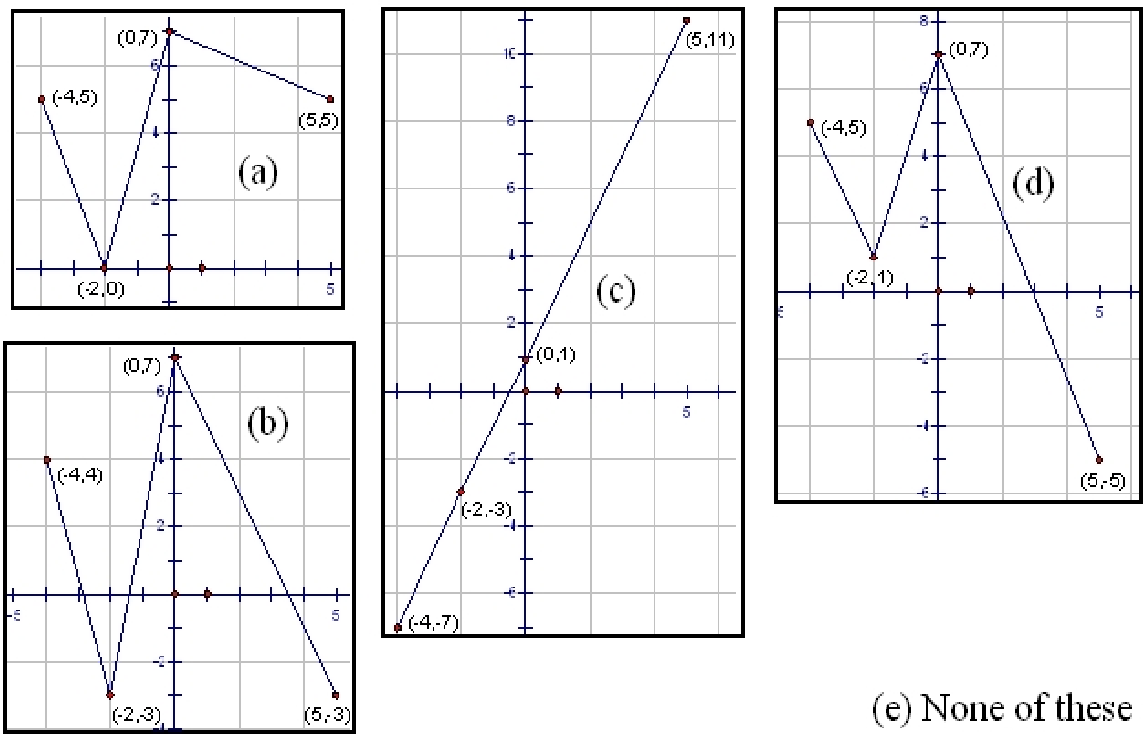
   Correct answer: **(a) 64**

   This is simply a trial-and-error problem – that is, for each potential answer, we simply add that test score to the sample and calculate the new mean and median. If the extra data point is 64, then our sample becomes {64, 72, 78, 81, 95}. Note that the "middle", or median, score is 78; also, the mean of these scores is $\frac{64 + 72 + 78 + 81 + 95}{5} = \frac{390}{5} = 78$. Thus, the mean and the median are equal. (None of the other choices, when added to the sample, cause the mean to be equal to the median.)

7.



The above picture shows the graph of some function $f(x)$. Which one of the following shows the graph of $2f(x) + 1$?



Correct answer: (d)

Observe that for each value of x, the corresponding value of y on the second graph – that is, $2f(x) + 1$ – must be one more than twice the corresponding y-value on the first graph. So, for example:

- From the graph $y = f(x)$, we see that $f(-4) = 2$. So, in the graph $y = 2f(x) + 1$, the y-value corresopnding to $x = -4$ should be $2(2) + 1 = 5$. This rules out (b) and (c) as possible correct answers.
- From the graph $y = f(x)$, we see that $f(-2) = 0$; therefore, we have $2f(-2) + 1 = 2(0) + 1 = 1$, which means the point (-2,1) should appear in the graph $y = 2f(x) + 1$. This rules out (a) as a correct answer; the only remaining possibilities are (d) and (e).

If we check x=0, 2 and 4, we see that in each case, the y-value corresponding to each of these x-values in graph (d) is one more than twice the y-value corresponding to each of these x-values in the graph of $f(x)$. Therefore, graph (d) is the graph of $2f(x) + 1$.

8. Let P be the set of prime factors of the integer 2004, and let F be the set of the prime factors of 2006. How many elements are in the union of P and F?

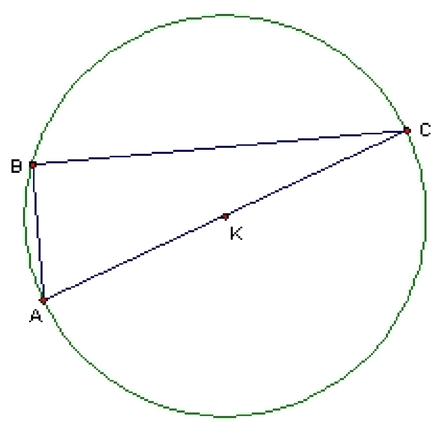
   (a) 1 (b) 3 (c) 4 (d) 5 (e) 6

   Correct answer: **(d) 5**

   The prime factorization of each of these numbers is as follows:

$$\begin{aligned} 2004 &= 2^2 \times 3 \times 167 \\ 2006 &= 2 \times 17 \times 59 \end{aligned}$$

   Thus, P={2,3,167} and F={2,17,59}, and so $P \cup F = \{2, 3, 167, 17, 59\}$.

9. 

   In the above diagram, the circle centered at K has radius 10, triangle ABC is a right triangle, AB=10, and AC is a diameter of the circle. Find the measure of the largest acute angle in triangle ABC.

   (a) 15° (b) 30° (c) 45° (d) 60° (e) 75°

   Correct answer: **(d)** 60°

   Since AC is a diameter of the circle, triangle ABC is a right triangle (with right angle B). Thus, AC=20, and AB=10 (given); since one leg of this right triangle is one half as long as the hypotenuse, this triangle is a $30° - 60° - 90°$ triangle.

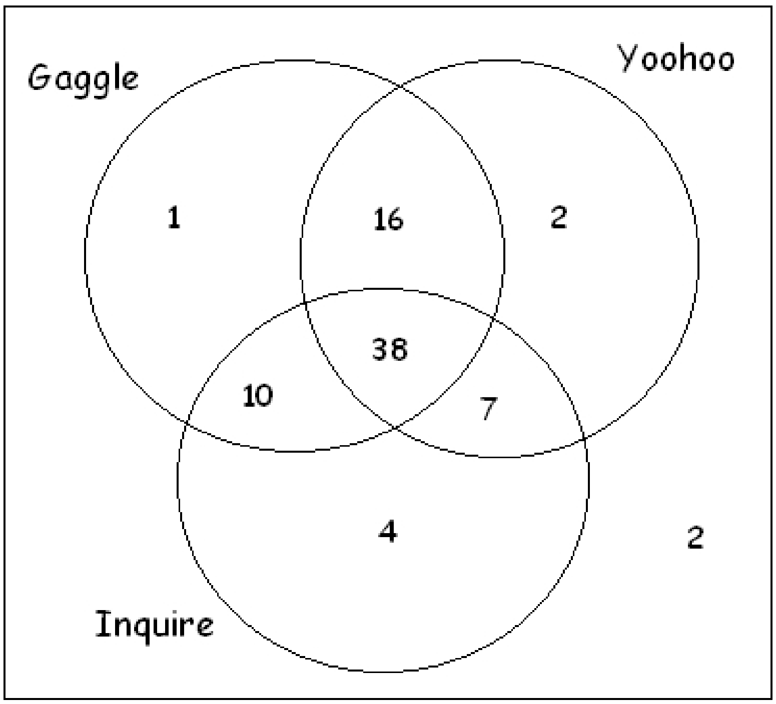
10. An internet research group chose 80 web sites at random, and searched for each of them on each of three search engines - Gaggle, Yoohoo, and Inquire - to see which sites had been found by each search engine. The results were as follows: 65 of the sites were found by Gaggle, 63 were found by Yoohoo, and 59 were found by Inquire. Further, 54 of the sites were found by both Gaggle and Yoohoo, 48 were found by both Gaggle and Inquire, and 45 were found by both Yoohoo and Inquire. There were 38 sites that were found by all three search engines.

How many of the chosen web sites were found by at least one of the three search engines?

(a) 2 (b) 40 (c) 65 (d) 78 (e) 187

Correct answer: **(d) 78**

This is shown by the following Venn diagram:



11. If P, Q and T are sets, then the statement "x belongs to $(P \cup Q) \cap (T \cap Q)$" is logically equivalent to which of the following?
    (a) x belongs to P and T
    (b) x belongs to T and Q
    (c) x belongs to P and Q but not T
    (d) x belongs to at least one of the three sets
    (e) x belongs to all three sets

    Correct answer: **(b) x belongs to T and Q**
    Notice that $T \cap Q \subseteq Q \subseteq P \cup Q$. So, if x belongs to $T \cap Q$, then it automatically belongs to both $P \cup Q$ and $T \cap Q$. Conversely, if x belongs to $(P \cup Q) \cap (T \cap Q)$, then clearly x belongs to $T \cap Q$. Thus, the statements "x belongs to $(P \cup Q) \cap (T \cap Q)$" and "$x$ belongs to T and Q"' are equivalent. (In other words, $(P \cup Q) \cap (T \cap Q) = T \cap Q$.)

12. A radio station is giving away concert tickets. A list of ten songs will be made available to listeners. Listeners are to call in their pick of three songs. The station will play a random selection of three songs from the list of ten. If a caller matches the three songs to those that are played on the radio (but not necessarily in the same order as played), the caller wins tickets to the concert. What is each caller's probability of winning the tickets?

(a) 1/1000 (b) 1/720 (c) 1/120 (d) 1/30 (e) 3/10

Correct answer: **(c) 1/120**

This is a *combinations* problem – we must count the number of possible unordered selections, without repetition, of three songs from ten. The number of such selection is given by the combinations formula:

$$\binom{10}{3} = \frac{10!}{3!7!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120.$$

Since there are 120 such combinations, a caller's probability of choosing the correct one is 1/120.

13. Solve for x:

$$2^{\log_2 e^{\ln 5^{\log_5 7^{\log_7 10^{\log(8x-3)}}}}} = 13$$

(a) $\dfrac{\log_2 13 + 3}{8}$ (b) 1 (c) 2 (d) e (e) $\dfrac{10^{13} + 3}{8}$

Correct answer: **(c) 2**

When working with a tower of exponents such as this one, we are to work from the top down. At each step, notice that we have an expression of the form $b^{\log_b k}$, which (by definition of logarithm base b) is equal to k. For example, at the top of the tower, we have the expression $10^{\log(8x-3)}$; since "log" is taken to mean the base 10 logarithm, this expression is simply equal to $8x - 3$. At the next step, then, we have $7^{\log_7(8x-3)}$ – which, again, is equal to $8x - 3$. This process continues all the way down the tower of exponents, until at the end we are left with $2^{\log_2(8x-3)}$, which is equal to $8x-3$. Thus, the original equation reduces to $8x - 3 = 13$, whose solution is $x = 2$.

14. Assume the following statement is true:

"If the watch costs less than 40 dollars, then Jim will buy it."

Now consider the following three statements:
I. If the watch costs more than 40 dollars, then Jim will not buy it.
II. If Jim will not buy the watch, then the watch costs at least 40 dollars.
III. If Jim will buy the watch, then the watch costs less than 40 dollars.

Which of these three statements must be true?

(a) I only

(b) I and III only

(c) II only

(d) II and III only

(e) None of these statements must be true

Correct answer: **(c) II only**
Statements I, II, and III are the inverse, contrapositive, and converse (respectively) of the original statement. Of these, only the contrapositive is logically equivalent to a root statement; therefore, only statement II need be true if the original statement is true.

15. On his 201st birthday, Jack Skellington decides to put \$1000 in a coffin to save for his 300th birthday party. He plans to save some money every year on his birthday, although in decreasing amounts. For his 202nd birthday, he puts another \$500 in the coffin and for his 203rd he puts in \$250. Jack likes the idea of putting in exactly half the amount he contributed the previous year. If he continues to do this on each birthday up to (and including) his 300th birthday, then how much money (in dollars) will he have saved for his 300th birthday party?

    (a) $2000((1/2)^{300})$ (b) $2000((1/2)^{100})$ (c) $300(1-(1/2)^{2000})$ (d) $1000(1-(1/2)^{200})$
    (e) $2000(1-(1/2)^{100})$

    Correct answer: **(e)** $2000(1-(1/2)^{100}$

    We are being to evalaute the geometric series

$$1000 + 1000\left(\frac{1}{2}\right) + 1000\left(\frac{1}{2}\right)^2 + 1000\left(\frac{1}{2}\right)^3 + \ldots + 1000\left(\frac{1}{2}\right)^{99}.$$

    The formula to evaluate such a series is

$$\sum_{k=0}^{n} ar^n = \frac{a(1-r^{n+1})}{1-r}.$$

    In this instance, we have constant factor $a = 1000$ and ratio $r = 1/2$, and so the value of this series is

$$\frac{1000(1-(1/2)^{99+1})}{1-(1/2)}.$$

    Rewrite the denominator as $1/2$, multiply by 2 on the top and bottom, and this simplifies to

$$2000(1-(1/2)^{100}).$$

16. How many times does the digit 1 appear in the base two (binary) representation of the decimal number 118?

    (a) 3 (b) 4 (c) 5 (d) 6 (e) 7

    Correct answer: **(c) 5**

    To find its binary representation, we must write 118 as a sum of powers of 2. The largest power of 2 less than 118 is 64, so the binary representation of 118 has a 1 in the 64's place. We continue this process as follows:

$$\begin{aligned} 118 &= 64 + 54 \\ &= 64 + 32 + 22 \\ &= 64 + 32 + 16 + 6 \\ &= 64 + 32 + 16 + 4 + 2. \end{aligned}$$

Thus, we can write

$$118 = 1 \cdot 64 + 1 \cdot 32 + 1 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1,$$

so the binary representation of 118 is

$$1110110_2.$$

17. If a, b, c and d are all positive real numbers, then which of the following is equivalent to $\log_d a \cdot \log_c b$?

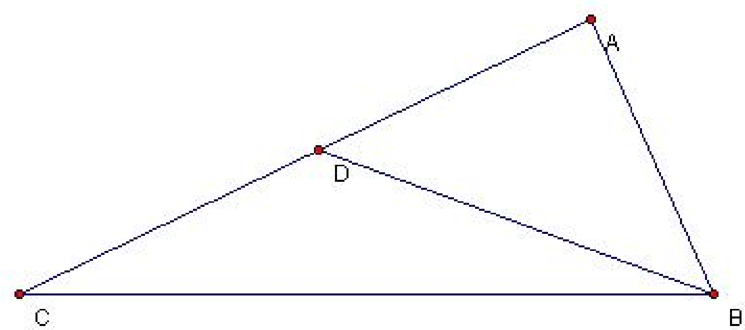
(a) $\log_a b \cdot \log_c d$ (b) $\log_a d \cdot \log_b c$ (c) $\log_c a \cdot \log_d b$ (d) $\log_c d \cdot \log_b a$ (e) $\log_d a \log_b a$

Correct answer: **(c)** $\log_c a \cdot \log_d b$

Use the change of base formula for logarithms:

$$\log_d a \cdot \log_c b = \frac{\log a}{\log d} \cdot \frac{\log b}{\log c} = \frac{\log a}{\log c} \cdot \frac{\log b}{\log d} = \log_c a \cdot \log_d b.$$

18. (First tiebreaker)

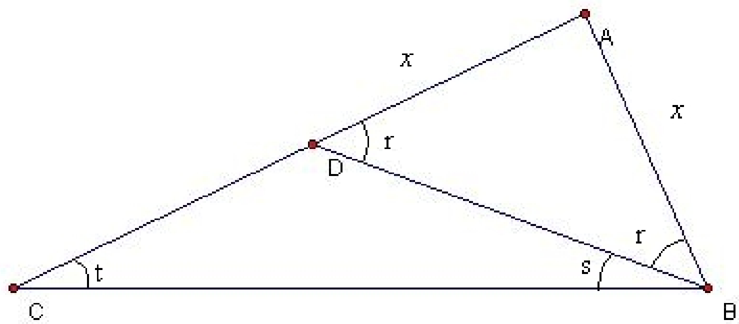


In the above diagram, $AB = AD$ and $m\angle ABC - m\angle ACB = 30°$. Find $m\angle CBD$.

(a) 15°
(b) 20°
(c) 22.5°
(d) 30°
(e) 45°

Correct answer: **(a) 15°**

Consider the following modified version of the original diagram:



We are given that triangle ABD is isosceles, which implies $m\angle ADB = m\angle ABD$, so we can use r for both. Also, $m\angle ABC - m\angle ACB = 30°$, which means $r + s - t = 30°$. We also know $m\angle CDB = 180° - r$, and so $s + t + (180° - r) = 180°$, which implies $r = s + t$.

Thus, we have $r + s - t = 30°$ and $r = s + t$; combining these two pieces of information, we have $(s + t) + s - t = 30°$; that is, $2s = 30°$, and so $s = 15°$. Thus, $m\angle CBD = 15°$.

19. (Second tiebreaker) Let K and L be points on the circle centered at the origin with radius 3. Both points are moving along the circle in the counter-clockwise direction; K is moving with rotational velocity $\pi/3$ and L is moving with rotational velocity $\pi/4$. If K is located at (3,0) and L is at $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$, then how many units of time will elapse before K and L occupy the same location ?

    (a) 2 (b) 3 (c) 4 (d) 5 (e) 6

    Correct answer: **(a) 2**

    Rotational velocity refers to the change in angle (radian measure) per unit of time. Initially point K is at (3,0), so its angular position is 0. Point L is initially at $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$, so its initial angle is $\arctan\left(\frac{3/2}{3\sqrt{3}/2}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. So, at time t, point K's angular position will be $0 + \frac{pi}{3} \cdot t$, and point L's angular position will be $\frac{\pi}{6} + \frac{\pi}{4} \cdot t$. Thus, the points will occupy the same location when $\frac{pi}{3} \cdot t = \frac{\pi}{6} + \frac{\pi}{4} \cdot t$. A bit of algebra gives us the solution $t = 2$.

20. (Third tiebreaker) Evaluate:

$$\left(2 + \cfrac{3}{4 + \cfrac{3}{4 + \cfrac{3}{4 + \ldots}}}\right)^2.$$

    (a) 6 (b) 7 (c) e^2 (d) 8 (e) 25

Correct answer: **(b) 7**

Let

$$x = 2 + \cfrac{3}{4 + \cfrac{3}{4 + \cfrac{3}{4 + \dots}}},$$

and observe that this can be rewritten as

$$x = 2 + \cfrac{3}{2 + \left(2 + \cfrac{3}{4 + \cfrac{3}{4 + \dots}}\right)}.$$

Now, the quantity inside the parentheses is just x again, so we now have an equation that can be solved for x:

$$\begin{aligned} x &= 2 + \frac{3}{2+x} \\ x(2+x) &= 2(2+x) + 3 \\ x^2 + 2x &= 4 + 2x + 3 \\ x^2 &= 7 \end{aligned}$$

Since x^2 is the quantity we are asked to evaluate, the answer is 7.