

Eastern Shore High School Mathematics Competition
 Individual Contest Solutions & Hints
 November 9, 2005

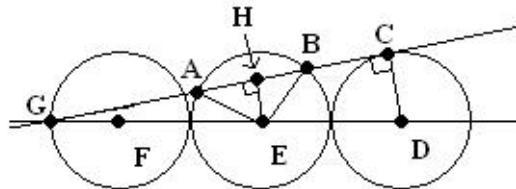
1. Bean-counting problem

The key to this problem is to observe that after each iteration of steps 1 and 2, the following have occurred:

- The total number of beans has been decreased by one, and
- The total number of white beans has been decreased by an even number (either 2 or 0).

Therefore, if you start with an odd number of white beans, then after each iteration – all the way down to the last bean – there will still be an odd number of white beans; in this case, the last bean must be a white one. Similarly, if you start with an even number of white beans, then (since 1 is not an even number) the last bean in the cup must be a black one.

2. Modify the given diagram by adding a few line segments, as shown here:



(Segments EH and CD are perpendicular to line GC , with point H lying on line GC .) Note that right triangles $\triangle GCD$ and $\triangle GHE$ are similar; this allows you to find EH in terms of CD (which is equal to r). Next consider right triangles $\triangle AHE$ and $\triangle BHE$. Since $AE = BE = r$ and you have EH (as a multiple of r), AH and BH follow by the Pythagorean Theorem. Your final answer should be $AB = \frac{8}{5}r$.

3. In the calculation below, each letter represents a digit. Find the digits represented by A, B and C.

$$\begin{array}{r}
 ABC \\
 \times ABC \\
 \hline
 DEFC \\
 CEBH \\
 EKKH \\
 \hline
 EAGFFC
 \end{array}$$

Some deductive reasoning helps in solving this problem. Things to notice:

- In the tens place of the product, note that we have $F + H = F$. This means $H = 0$, and so the product $B \times ABC$ is a multiple of 10. Therefore, we must have $B = 0$, $C = 0$, or $C = 5$ (in which case B must be even).
- Note also that C is the first digit of $B \times ABC$; therefore, $C = 0$ doesn't make sense, so $C = 5$ (which implies B is even).

- The units digit of $A \times ABC$ is also 0 (since $H = 0$), so $A \times ABC$ is also a multiple of 10. Therefore, A is even.

These are a few observations to help you get started; a few more, along with some trial-and-error, are also necessary to eventually reach a unique valid solution.

Solution: $A = 6, B = 8, C = 5$.

4. (Note: The team exam used at the 2005 competition included an error in the definition of “one to one correspondence”. The solution given below corresponds to the *correct* definition of one to one correspondence, rather than the erroneous one given on the contest exam. For details, go to the HSMC web page and download the 2005 team contest exam.)

Definition: A “one to one correspondence” between sets A and B is a mapping from set A into set B such that each element of set A is mapped to exactly one element of set B and each element of set B has *exactly one* element of set A mapped to it.

Instructions: For each of the following pairs of sets, do one of the following:

- Give an example of a one to one correspondence between A and B .
...OR...
- Explain why no such mapping from A into B exists.

(a) $A = \{a, b, c, d\}, B = \{2, 3, 5, 6\}$

(b) $A = \{a, b, c\}, B = \{2, 3, 5, 6\}$

(c) $A = \{a, b, c, d\}, B = \{2, 3, 5\}$

(d) $A =$ the set of all integers, $B =$ the set of all even integers

The main idea of parts (a) through (c) is that for any two *finite* sets A and B , a one-to-one correspondence between the sets exists if and only if A and B have the same number of elements.

- (a) A mapping such as $a \rightarrow 2, b \rightarrow 3, c \rightarrow 5, d \rightarrow 6$ is a one-to-one correspondence between these two sets.
- (b) In this case, B has more elements than A . Therefore, if each element of A is mapped to exactly one element of B , then (at least) one element of B will have nothing mapped to it. Therefore, no one-to-one correspondence exists between these two sets.
- (c) In this case, A has more elements than B . Therefore, if each element of A is mapped to an element of B , then (at least) one element of B will have more than one element of A mapped to it. Therefore, no one-to-one correspondence exists between these two sets.
- (d) The mapping $n \rightarrow 2n$ (where $n \in A$) is a one-to-one correspondence between A and B .