Eastern Shore High School Mathematics Competition
November 9, 2005
Team Contest

Answer as many questions as possible in the time provided. To receive full credit for a correct solution, show all work and provide a clearly written explanation. Solutions will be judged based on correctness, completeness and clarity. (Little credit, if any, will be given for a solution consisting of just a number or a single sentence.)

All work and answers must be written on the provided sheets of plain white paper. Use only one side of each sheet of paper, and start each new problem on a new sheet of paper.

1. Your team will be provided with a cup, several black beans, and several white beans.

   (a) Put five white beans and ten black beans in the cup, and repeat the following process until exactly one bean remains in the cup at the end of Step 2.

   - Step 1: Randomly select two beans from the cup.
   - Step 2: If both beans are white, put them aside and add another black bean to the cup. Otherwise, one of the two beans you selected is black. From the two beans you selected, put one black bean aside, and return the other bean (white or black!) to the cup.

   What is the color of the last bean in the cup?

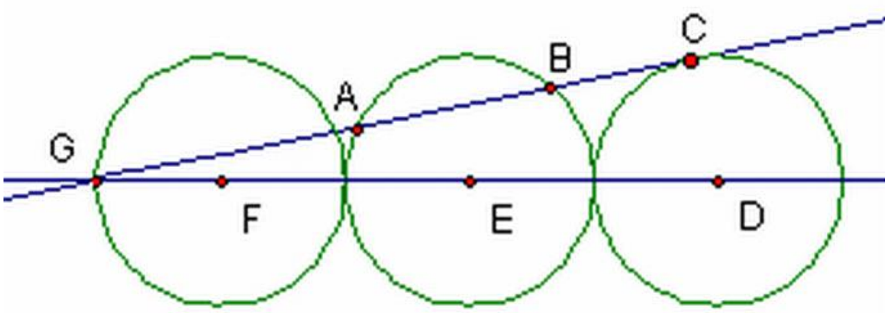
   (b) Start with six white beans and *any* number of black beans in the cup, and again repeat the above process until exactly one bean remains in the cup at the end of Step 2.

   What is the color of the last bean in the cup this time?

   (c) If you repeated the process as above, but starting with 2005 white beans and 2005 black beans in a (very large!) cup, what would be the color of the last bean in the cup? Justify your answer. *(You are encouraged to conduct experiments similar to parts (a) and (b) to understand the problem better; however, the best justification would not rely on such examples for evidence.)*

   (d) In general, what would happen if you started with n white beans and m black beans?

2. In the figure shown below, each circle has the same radius, r. The centers of these circles, points D, E, and F, are collinear. Point G lies on line DF and also on the circle centered at point F. Point C lies on the circle centered at point D, and line GC is tangent to this circle. Points A and B are the intersections of line GC with the circle centered at point E. Find the length of segment AB in terms of r.



3. In the calculation below, each letter represents a digit. Find the digits represented by A, B and C.

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     ABC
   x ABC
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    DEFC
   CEBH
  EKKH
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  EAGFFC
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4. A "one to one correspondence" between sets A and B is a mapping from set A into set B such that each element of set A is mapped to exactly one element of set B and each element of set B has *exactly one** element of set A mapped to it. For example: if $A = \{1, 2, 3\}$ and $B = \{3, 6, 9\}$, then the mapping:

$$\begin{array}{ccc} 1 & \rightarrow & 3 \\ 2 & \rightarrow & 6 \\ 3 & \rightarrow & 9 \end{array}$$

is a one to one correspondence between A and B, since under this mapping each element of set A is mapped to exactly one element of set B and each element of set B has an element of set A mapped to it. (The mapping shown above could also be described as the *function* $f(x) = 3x$, where x is an element of A and $3x$ is the corresponding element of B.)

Instructions: For each of the following pairs of sets, do one of the following:

- Give an example of a one to one correspondence between A and B.
  ...OR...
- Explain why no such mapping from A into B exists.

(a) $A = \{a, b, c, d\}$, $B = \{2, 3, 5, 6\}$
(b) $A = \{a, b, c\}$, $B = \{2, 3, 5, 6\}$
(c) $A = \{a, b, c, d\}$, $B = \{2, 3, 5\}$
(d) A = the set of all integers, B = the set of all even integers

* There was an error in the wording of this question which, unfortunately, was not caught before the date of the contest. The incorrectly worded question's definition of a one to one correspondence included the condition "each element of set B has *at least* one element of set A mapped to it". (The correct definition, with "at least" replaced by "exactly one", is given in the above instructions on this belatedly corrected version of the team contest exam.)

As a result of this error, some teams answered the question based on the definition given in the instructions, while others – who already knew the definition of a one to one correspondence – answered the question based on the correct definition. In order to avoid penalizing any team for our error, we decided to grade each team's work based on the specific definition of "one to one correspondence" that team elected to use. As long as a team used either the correct definition or the definition given in the exam instructions, *and* they applied this definition correctly and consistently throughout their written solution, then they received credit for a correct solution to the problem.

The team solutions – those provided after the contest and also those linked on the HSMC web page – are based on the correct definition of a one to one correspondence.