

Eastern Shore High School Mathematics Competition  
Individual Contest Solutions  
November 9, 2005

1. Which of the following is a solution to the equation  $\frac{x(x-3)}{(x-1)(x-2)} = -9$ ?

(a) 0.5    (b) 1.5    (c) 2.5    (d) 3.5    (e) None of these

Answer: (e) None of these

This can be verified by simply substituting each of the choices for  $x$  and evaluating the expression. In each case, its value is *not* -9.

2. Which of the following is equivalent to  $x - 2(x + 3(x - 4(x - y)))$ ?

(a)  $-2x - y$     (b)  $17x - 24y$     (c)  $33x + 12y$     (d)  $x^3 + x^2 - 10x + 4y$     (e)  $-24x^4y$

Answer: (b)  $17x - 24y$

This problem is an easy one as long as one remembers to observe the correct *order of operations*. In particular, quantities inside parentheses are to be evaluated first; in the case of nested parentheses, work from the inside out:

$$\begin{aligned}x - 2(x + 3(x - 4(x - y))) &= x - 2(x + 3(x - 4x + 4y)) \\ &= x - 2(x + 3(-3x + 4y)) \\ &= x - 2(x + (-9x + 12y)) \\ &= x - 2(-8x + 12y) \\ &= x - (-16x + 24y) \\ &= 17x + 24y\end{aligned}$$

3. On three math tests, Nathan's mean score was 80, while on five reading tests, his mean score was 90. What was Nathan's mean score over all eight tests?

(a) 21.25    (b) 83.75    (c) 85    (d) 86.25    (e) Cannot be determined

Answer: (d) 86.25

The sum of Nathan's three math test scores was  $3 \times 80 = 240$ , and the sum of his three reading test scores was  $5 \times 90 = 450$ . Therefore the sum of his eight test scores was  $450 + 240 = 690$ , which implies that his mean score over the eight tests was  $\frac{690}{8} = 86\frac{1}{4}$ .

4. You are given a regular pentagon in the  $xy$  plane. The endpoints of one of its sides have coordinates (0,3) and (3,7). What is the perimeter of the pentagon?

(a) 4    (b) 5    (c) 20    (d) 25    (e) 125

Answer: (d) 25

Since it's a regular pentagon, each of its five sides has the same length. From the endpoints of one side, we can find (using the distance formula, or equivalently the Pythagorean Theorem) that its side length is  $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$ . Therefore, the pentagon's perimeter is  $5 \times 5 = 25$ .

5. Let  $x$  and  $y$  be numbers such that their sum is equal to their product. Which of the following statements *must* be true?

I.  $x \neq 1$  and  $y \neq 1$ .      II.  $\frac{y}{y-1} + \frac{x}{x-1} = xy$ .      III.  $(x-1)(y-1) = 1$ .

- (a) I only
- (b) II only
- (c) I and II but not III
- (d) II and III but not I
- (e) I, II and III

Answer: (e) I, II and III are true.

We are given that  $x + y = xy$ . Right away we see that  $x \neq 1$ , since if  $x$  were equal to 1 then we'd have  $1 + y = y$ , which is clearly impossible. (Similarly,  $y \neq 1$ .) Therefore, statement I is true.

If we solve for  $y$  in terms of  $x$ , we find that  $y = \frac{x}{x-1}$ :

$$\begin{aligned} xy &= x + y \\ xy - y &= x \\ y(x - 1) &= x \\ y &= \frac{x}{x - 1} \end{aligned}$$

Similarly,  $x = \frac{y}{y-1}$ . Therefore, statement II is equivalent to  $x + y = xy$ , so II is true.

Finally, we can rewrite the left-hand side of statement III as follows:

$$\begin{aligned} (x - 1)(y - 1) &= xy - x - y + 1 \\ &= xy - (x + y) + 1 \\ &= 1, \text{ since } xy = x + y. \end{aligned}$$

Thus, the assumption  $x + y = xy$  implies I, II and III.

6. Evaluate the expression:  $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}$

- (a)  $\frac{1 + \sqrt{5}}{2}$
- (b) 2
- (c)  $e$
- (d) 4
- (e)  $\infty$

Answer: (b) 2

Let  $x$  denote the value of the given expression, and notice that  $x = \sqrt{2+x}$ . That is,

$$\left\{ \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}} \right\} = \sqrt{2 + \left\{ \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}} \right\}}.$$

To evaluate the expression, then, we must solve for  $x$ :

$$\begin{aligned} x &= \sqrt{2+x} \\ x^2 &= 2+x \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0, \text{ so } x = -1 \text{ or } x = 2. \end{aligned}$$

Clearly  $x > 0$ , so the correct answer is  $x = 2$ .

7. If  $a, b, c$  and  $d$  are positive integers, compute the numerical value of

$$\log_{10} \left( \frac{2a}{b} \right) - \log_{10} \left( \frac{c}{2b} \right) + \log_{10} \left( \frac{5c}{d} \right) - \log_{10} \left( \frac{a}{5d} \right) + \log_{10} 1.$$

- (a) 0    (b) 2    (c) 3    (d) 6    (e) None of these

Answer: (b) 2

Using properties of logarithms:

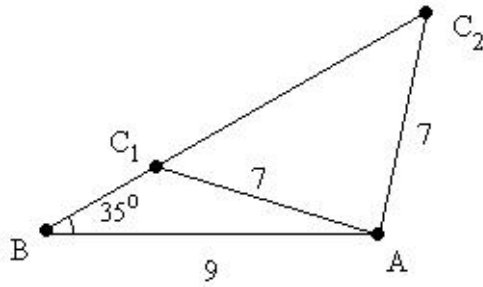
$$\begin{aligned} \log_{10} \left( \frac{2a}{b} \right) - \log_{10} \left( \frac{c}{2b} \right) + \log_{10} \left( \frac{5c}{d} \right) - \log_{10} \left( \frac{a}{5d} \right) + \log_{10} 1 \\ &= \log_{10} \left( \frac{\left( \frac{2a}{b} \right) \left( \frac{5c}{d} \right)}{\left( \frac{c}{2b} \right) \left( \frac{a}{5d} \right)} \right) + 0 \\ &= \log_{10} \left( \left( \frac{2a}{b} \right) \left( \frac{2b}{c} \right) \left( \frac{5c}{d} \right) \left( \frac{5d}{a} \right) \right) \\ &= \log_{10} \left( \frac{100abcd}{abcd} \right) \\ &= \log_{10} 100 = 2. \end{aligned}$$

8. How many non-congruent triangles  $ABC$  exist such that  $b = 7$ ,  $c = 9$  and  $m\angle ABC = 35^\circ$ ? (Assume  $b$  is the length of the side opposite vertex  $B$  and  $c$  is the length of the side opposite vertex  $C$ .)

- (a) 0    (b) 1    (c) 2    (d) Infinitely many    (e) Cannot be determined

Answer: (c) 2

Note: the side-angle-side congruence property of triangles doesn't apply here, since  $\angle ABC$  is not the included angle of the sides whose lengths are given. The following diagram demonstrates that two different triangles satisfying the given conditions are possible:

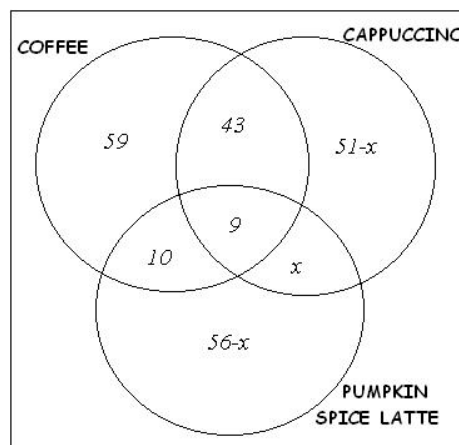


Triangles  $ABC_1$  and  $ABC_2$  clearly are not congruent, but both triangles satisfy  $b = 7$ ,  $c = 9$  and  $m\angle ABC = 35^\circ$ . However, there is no other point on the line containing  $C_1$  and  $C_2$  that would also be 7 units from point A; therefore, these two triangles are the only ones satisfying the specified conditions.

9. The local coffee shop is serving coffee, cappuccino and pumpkin spice latte. On Friday, 200 people purchased at least one of these drinks. If 121 of these people purchased coffee, 103 purchased cappuccino, 75 purchased pumpkin spice latte, 52 purchased coffee and cappuccino, 19 purchased coffee and pumpkin spice latte, and 9 people purchased all three drinks, how many purchased cappuccino and pumpkin spice latte?
- (a) 28
  - (b) 37
  - (c) 141
  - (d) There is not enough information provided to answer this question.
  - (e) None of the above.

Answer: (b) 37

This is a standard 3-circle Venn diagram problem. Use one circle for the set of people who purchased coffee, one for the set of people who purchases cappuccino, and one for the set of people who purchased latte. The following diagram summarizes the given data, with  $x$  denoting the number of people who purchased cappuccino and latte.



The numbers in the diagram must add up to 200; this leads to the equation  $228 - x = 200$ , hence  $x = 28$ . Therefore, there are  $9 + 28 = 37$  people who purchased cappuccino and pumpkin spice latte.

10. If  $f(x) = x^{-3}$ , find the value of  $f(f(f(f(f(f(2))))))$ .

- (a) 1      (b)  $\frac{1}{729}$       (c) 729      (d)  $729^{-3}$       (e) None of these

Answer: (e) None of these.

In general,  $f(f(f(f(f(f(x)))))) = (((((x^{-3})^{-3})^{-3})^{-3})^{-3})^{-3})^{-3}$ , which (by repeated use of the property  $(a^b)^c = a^{bc}$ ) is equal to  $x^{729}$ . Therefore,  $f(f(f(f(f(f(2)))))) = 2^{729}$ , which is somewhat larger than any of the available choices.

11. The function  $f$  is given in the following table: 

x	1	2	3	4	5
f(x)	5	4	1	2	3

.

If  $u_1 = 3$  and  $u_{n+1} = f(u_n)$  for  $n > 0$ , what is the value of  $u_{2005}$ ?

- (a) 1      (b) 2      (c) 3      (d) 4      (e) 5

Answer: (c) 3

Start by finding the value of  $u_n$  for  $n = 1, 2, 3, \dots$ , and look for repetition:

$$\begin{aligned} u_1 &= 3 \\ u_2 &= f(3) = 1 \\ u_3 &= f(1) = 5 \\ u_4 &= f(5) = 3 \end{aligned}$$

This indicates a *cycle* of length 3 – that is, any time  $n$  is one more than a multiple of 3,  $u_n$  will be equal to 3. (If you don't yet see this, list out the values of  $u_n$  up to  $n = 10$ , and see what pattern emerges.) Since 2005 is one more than a multiple of 3, it follows that  $u_{2005} = 3$ .

12. Adam's age equals Beth's age plus the cube root of Chris's age. Beth's age is 14 more than the sum of Chris's age and the cube root of Adam's age. Chris's age equals the cube root of Adam's age plus the square root of Beth's age. What is Adam's age? (Assume that each person's age is a whole number.)

- (a) 1      (b) 8      (c) 27      (d) 64      (e) None of these

Answer: (c) 27

If we let  $a, b, c$  denote Adam's, Beth's and Chris's ages, respectively, then we have the following system of equations:

$$\begin{aligned} a &= b + \sqrt[3]{c} \\ b &= 14 + c + \sqrt[3]{a} \\ c &= \sqrt[3]{a} + \sqrt{b}. \end{aligned}$$

At this point, we can use trial and error, substituting each available choice for  $a$  to see if a valid

solution is reached. For example, try  $a = 8$ ; this gives us:

$$\begin{aligned}8 &= b + \sqrt[3]{c} \\ b &= 14 + c + \sqrt[3]{8} \\ c &= \sqrt[3]{8} + \sqrt{b},\end{aligned}$$

which simplifies to

$$\begin{aligned}8 &= b + \sqrt[3]{c} \\ b &= 16 + c \\ c &= 2 + \sqrt{b}.\end{aligned}$$

The last two equations in this system can be combined into the single equation  $b = 18 + \sqrt{b}$ , which has no integer solution; therefore,  $a = 8$  is not the correct solution. ( $a = 1$  and  $a = 64$  can be similarly ruled out.) However, for  $a = 27$ , we get the system

$$\begin{aligned}27 &= b + \sqrt[3]{c} \\ b &= 17 + c \\ c &= 3 + \sqrt{b}.\end{aligned}$$

This time, the last two equations give us  $b = 20 + \sqrt{b}$ , which has the solution  $b = 25$ . Substituting  $a = 27, b = 25$  into the system gives us the simultaneous solution  $(a, b, c) = (27, 25, 8)$ . Therefore, Adam must be 27 years old.

13. Points K and L move with constant rotational velocities in a counterclockwise direction around a circle with center at the origin and radius 4. If the rotational velocities of K and L are  $\frac{\pi}{2}$  and  $\frac{\pi}{4}$  respectively, and if both are located at  $(4,0)$  at time  $t = 0$ , how many units of time will pass before the two points coincide again?
- (a) 2      (b) 4      (c) 8      (d) 12      (e) None of these

Answer: (c) 8

Solution: The rotational velocity of each particle is defined as the distance travelled per unit of time divided by the radius, or (more simply) as the number of radians traversed per unit of time. Since there are  $2\pi$  radians in a circle (regardless of its radius), the two particles will coincide whenever the difference between their distances travelled (measured in radians) is a multiple of  $2\pi$ . So, we must find the least positive solution of the equation  $\frac{\pi}{2}t - \frac{\pi}{4}t = 2\pi n$ , where  $n$  is an integer:

$$\begin{aligned}\frac{\pi t}{2} - \frac{\pi t}{4} &= 2\pi n \\ \frac{\pi t}{4} &= 2\pi n \\ \pi t &= 8\pi n \\ t &= 8n\end{aligned}$$

It follows that the *least* positive solution for  $t$  occurs when  $n = 1$ ; that is,  $t = 8$ . Therefore, the particles again coincide after 8 units of time.

14. Suppose 60% of the people at a concert are under 24 years old. Which of the following *must* be true about the entire concert audience?
- (a) Their mean age is less than 24.
  - (b) Their median age is less than 24.
  - (c) Their mode age is less than 24.
  - (d) More than one of the above statements must be true.
  - (e) None of the above statements must be true.

Answer: (b) Their median age is less than 24.

The median of a set of data is defined as the number that is greater than or equal to exactly one half of the numbers in the set. Therefore, any number that is greater than 60% of the numbers in a set must be greater than the median. It follows that the median age of this audience is less than 24. This rules out (e) as the correct answer, and leaves us with only (b) and (d) as possibilities.

To rule out choice (d), we must show that neither (a) nor (c) is necessarily true. One way to do this is to consider a hypothetical audience of ten people with ages 18, 19, 20, 21, 22, 23, 30, 30, 30, 30. (This is just one example among infinitely many that could be used here.) Clearly 60% of these people are under 24 years old; however, their mean age is 24.3 and their mode age is 30. Thus, neither (a) nor (c) must be true.

15. Suppose  $a^{10} = m$  and  $\log_2 m = 10$ . Which of the following is true?
- (a)  $a^{10} = 100$
  - (b)  $a = m$
  - (c)  $m = 10$
  - (d)  $a = 2^{10}$
  - (e) None of these

Answer: (e) None of these

If  $\log_2 m = 10$ , then  $m = 2^{10}$ . Since  $a^{10} = m$ , it follows that  $a = \pm 2$ . None of the available choices corresponds to these values of  $a$  and  $m$ , so the correct answer is (e).

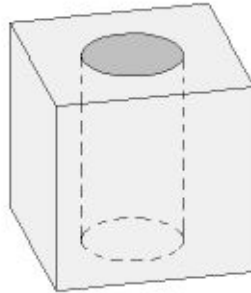
16. Sam and Jan are two of the ten students in a club. Two students from this club will be chosen at random to go on a field trip. What is the probability that Sam will be chosen to go on the field trip, but Jan will not be chosen?
- (a)  $2/45$
  - (b)  $8/45$
  - (c)  $9/45$
  - (d)  $16/45$
  - (e)  $17/45$

Answer: (b)  $8/45$

One method of solution is by using combinations. There are  $\binom{10}{2} = \frac{10 \times 9}{2 \times 1} = 45$  combinations of 2 students from 10 that can be chosen for the field trip. (Note that the denominator of each choice is 45; this was a hint!) Of these 45 pairs of students, there are exactly eight that consist of Sam and a student other than Jan (that is, one pair for each of the *other* eight students in the class). Therefore, the probability of choosing one of these pairs of students is  $8/45$ .

17. A cylindrical hole of radius 3 is bored through a cube of side length 10, as illustrated in the diagram below. Find the surface area of the resulting solid.

- (a)  $600 - 18\pi$     (b)  $1000 - 90\pi$     (c)  $1000 - 78\pi$     (d)  $600 + 42\pi$     (e)  $600 + 78\pi$

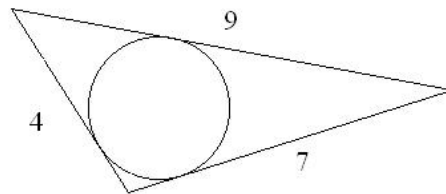


Answer: (d)  $600 + 42\pi$

The surface area of the object described will be the surface area of the cube, *minus* the combined area of the two circular bases of the cylindrical hole, *plus* the interior surface of the hole (whose area is equal to the lateral surface area of a cylinder, which is given by the formula  $L = 2\pi rh$ ).

This gives us  $A = 6(10)^2 - 2(\pi(3)^2) + 2\pi(3)(10) = 600 - 18\pi + 60\pi = 600 + 42\pi$ .

18. (First tiebreaker) Given a triangle with sides of length 4, 7 and 9 and an inscribed circle (as shown in the diagram below), what is the area of the circle?



Hint: The area of a triangle in terms of the lengths of its sides is given by Heron's formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where } s = \frac{1}{2}(a+b+c).$$

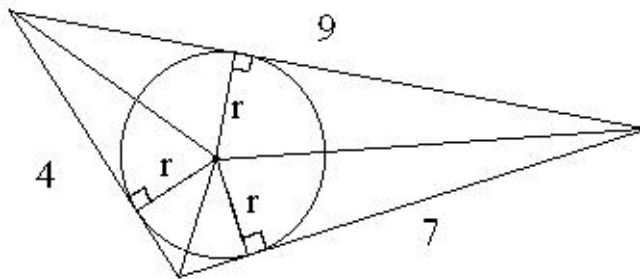
- (a)  $\frac{16\pi}{9}$     (b)  $\frac{9\pi}{5}$     (c)  $\sqrt{45}$     (d)  $\sqrt{180}$     (e)  $\frac{100\pi}{9}$

Solution: (b)  $\frac{9\pi}{5}$

To find the circle's area, we must first find its radius. First, use Heron's formula to find the area of the triangle:  $s = \frac{4+7+9}{2} = 10$ , so

$$A = \sqrt{10(10-4)(10-7)(10-9)} = \sqrt{10(1)(3)(6)} = \sqrt{180}.$$

Now, consider the following diagram:



We see that triangle ABC can be divided into three triangles, each with altitude  $r$  (the radius of the inscribed circle) and with bases 4, 7, and 9. The sum of the areas of these three triangles is

$$\begin{aligned}
 A &= \frac{1}{2}(r)(4) + \frac{1}{2}(r)(7) + \frac{1}{2}(r)(9) \\
 &= \frac{r}{2}(4 + 7 + 9) \\
 &= \frac{20r}{2} = 10r, \text{ which (by Heron's formula) equals } \sqrt{180}.
 \end{aligned}$$

Therefore,  $r = \frac{\sqrt{180}}{10}$ , so the area of the inscribed circle is

$$\pi r^2 = \pi \left( \frac{\sqrt{180}}{10} \right)^2 = \frac{180\pi}{100} = \frac{9\pi}{5}.$$

19. (Second tiebreaker) At Marvadel High School, some of the students in the marching band are in the math club, and some of the students in the math club are on the football team. However, since the marching band plays during football games, none of the students in the marching band are on the football team. Based on these assumptions, which of the following three statements *could* be true?

- I. Every student on the football team is in the math club.
- II. Every student in the math club is in the marching band.
- III. Some students in the math club are neither in the marching band nor on the football team.

- (a) Statements I and II, but not III
- (b) Statements I and III, but not II
- (c) Statements II and III, but not I
- (d) All three statements could be true
- (e) None of the statements could be true

Answer: (b) Statements I and III, but not II, could be true.

I: The *only* stated restriction on football players is that they can't be in the marching band; therefore it is possible that all of them are in the math club. (While there are marching band members in the math club too, this does not lead to a contradiction; it would just indicate that the football players constitute a proper subset of the math club.)

II: We are given that some of the students in the math club are on the football team. These students, who are in the math club, cannot be in the marching band, due to the assumption that no student is

both in marching band and on the football team. This makes it impossible for *every* student in the math club to be in the marching band; therefore statement II *cannot* be true.

III: Nothing in our assumptions requires every student in the math club to be in any other organization; therefore, it is possible for some students in the math club to be neither in the marching band nor on the football team.

20. (Third tiebreaker) What is the greatest prime number less than 900?

- (a) 891    (b) 893    (c) 897    (d) 899    (e) None of these

Answer: (e) None of these

Check each available choice for prime factors. It turns out that 891 and 897 are both divisible by 3,  $893 = 19 \times 47$ , and  $899 = 31 \times 29$ . Therefore, none of these is a prime number. (The greatest prime number less than 900 is actually 887.)