Eastern Shore High School Mathematics Competition
2004 Team Contest Solutions

1. Peg board game problem.

The table should be completed as follows:

| Number of pegs of each color: | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Minimum number of moves needed to complete the game: | 3 | 8 | 15 | 24 |

Explanations:

- Optimal Strategy Explanation: Move 1 blue peg, then 2 red pegs, then 3 blue pegs, ... then all of one color, then all of the other, and all of the first again. Now, move all but one of the other, all but two, all but three, ..., until you move the final peg. Total number of moves:

$$\begin{aligned} & (1+2+3+\ldots+x)+x+(x+\ldots+3+2+1) \\ = \;& 2(1+2+3+\ldots+x)+x \\ = \;& 2\frac{x(x+1)}{2}+x = x^2+2x. \end{aligned}$$

- Total Distance Moved Explanation: Each peg must move $x+1$ spaces through a combination of steps and jumps, for a total of 2x(x + 1) total spaces moved. Since each step moves a peg one space and each jump moves a peg two spaces, the total number of moves is the number of spaces minus the number of jumps. Each pair of pegs of opposing colors requires exactly one jump to swap sides. For each blue peg, then, there will be exactly $x$ jumps (one for each red peg). Therefore, there are exactly $x^2$ jumps. So the total number of moves is $2x(x+1)-x^2$, or $x^2+2x$ moves.
- Pattern Recognition and Induction from four examples: By the data we've collected, the pattern looks quadratic, so we could fit a quadratic polynomial to these four (x, y) pairs. The result is $y = x^2+2x$.

2. Formal dinner party problem.

Here is one possible method of finding the solution. We will denote each man at the party by the first letter of his name, for example, "A" for Alan.

We will write "$X \to Y$" to indicate that man X gave flowers to the date of man Y; for example, "$A \to B$" indicates that Adam gave flowers to Barry's date. Using this notation, we may translate each piece of information given in the problem as follows:

(a) $B \to \_\_\_ \to D$

(b) $A \to B$

(c) $F \to \_\_\_ \to A$

(d) $C \to \_\_\_ \to E$

(e) $D \to \_\_\_ \to \_\_\_ \to G$

(Here we use ___ to indicate a currently unknown person.) We can fit statements (a), (b), (c) and (e) together into the following:

$$A \to B \to \_\_\_ \to D \to \_\_\_ \to F \to G \to A.$$

Now, statement (d) can only fit with the others if $C \to D \to E$; thus, we have

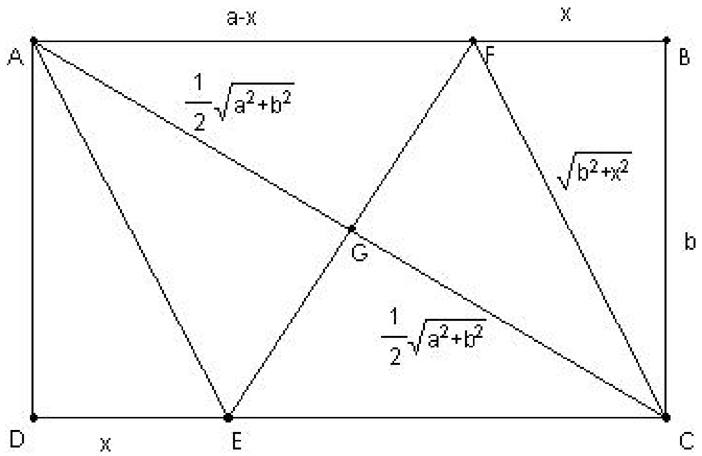
$$A \to B \to C \to D \to E \to F \to G \to A.$$

Therefore, $E \to F$; that is, Earl gave flowers to Farrell's date.

3. Rhombus problem.

We will solve the general case; the solution to the first part of the problem follows if we substitute $a = 16$ and $b = 12$.

To complete this problem, one must be aware that the diagonals of any rhombus are perpendicular line segments which bisect each other. With this in mind, we draw the second diagonal of rhombus AFCE and label the intersection of the two diagonals G (see diagram below). Thus, triangles $\triangle\ AGF$ and $\triangle\ FGC$ are congruent right triangles. Note also that AC is the hypotenuse of $\triangle\ ABC$; therefore, by the Pythagorean Theorem, $AC = \sqrt{a^2 + b^2}$, and it follows that $AG = GC = \frac{1}{2}\sqrt{a^2 + b^2}$.




Let BF=$x$; then AF=$a - x$. Since $\triangle\ AGF \cong \triangle\ FGC$, it follows that $a - x = \sqrt{b^2 + x^2}$, and so (solving for $x$) we have $x = \dfrac{a^2 - b^2}{2a}$. By the Pythagorean Theorem, $FG^2 + GC^2 = FC^2$; if we substitute the values shown in the above diagram and and solve for $FG$, we find that $FG = \dfrac{b}{2a}\sqrt{a^2 + b^2}$. Therefore, since $EF = 2FG$, the general solution of the problem is $\boldsymbol{EF = \frac{b}{a}\sqrt{a^2 + b^2}}$.

It follows that the solution for part (a) is $EF = \dfrac{12}{16}\sqrt{12^2 + 16^2} = \dfrac{3}{4}(20) = 15$.

4. Purple number problem.

Solutions:

(a) The smallest purple number greater than 2004 is 2040.
(b) The smallest purple number is 103.
(c) The largest purple number under 10,000 is 9130.

Comments:

- For part (b), once one has ruled out the possibility of a two-digit purple number, it's not hard to find the smallest three-digit purple number of 103. (Simply notice that $1^2 + 3^2 = 10$.)

  To rule out two digit numbers, one may reason as follows: assume that there is a two-digit purple number, xy. (Here x and y are digits; x is in the tens place, y in the ones place.) Then we must have $x^2 + y^2 = 10x + y$, or $x^2 - 10x = y - y^2$. Solving for $x$ (by completing the square or using the quadratic formula), we have $x = 5 \pm \sqrt{25 + y - y^2}$. However, the only integer values of $y$ for which $\sqrt{25 + y - y^2}$ is an integer are $y = 0$ and $y = 1$; in each of these cases, $\sqrt{25 + y - y^2} = 5$, so $x = 5 \pm 5 = 0$ or 10, neither of which can be the first digit of a two-digit number. Therefore, there is no two-digit purple number, and it follows that 103 is the smallest purple number.

- To find the largest purple number less than 10,000, one should consider four-digit numbers. It is logical to start with the possibility 99xy, then work downward to 98xy, 97xy, etc.

  As an example, we will show why 99xy cannot be purple number. For such a number to be purple, we must have $9^2 + 9^2 + x^2 + y^2 = 99$. However, $9^2 + 9^2 = 162 > 99$, so the sum of the squares of all four digits must be greater than the two-digit number formed by the first two digits. A similar argument rules out all four-digit numbers beginning with 98, 97, 96, 95 or 94 from being purple.

  If a purple number is of the form 93xy, then we must have $9^2 + 3^2 + x^2 + y^2 = 93$, which simplifies to $x^2 + y^2 = 3$, which has no integer solution (x,y). Similarly, 92xy is purple if and only if $x^2 + y^2 = 7$, which is also impossible for integer values of $x$ and $y$. However, the number 91xy is purple if and only if $x^2 + y^2 = 9$, which is true when (x,y) = (0,3) or (3,0). Therefore, both 9103 and 9130 are purple numbers, and so the largest purple number less than 10,000 is 9130.

5. Infinite product problem.

   The solution to the problem is $\frac{1}{1-x}$.

   In general, if $|r| < 1$, then

$$1 + r + r^2 + r^3 + ... + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

   Therefore,

$$1 + x + x^2 + x^3 + ... + x^9 = \frac{1 - x^{10}}{1 - x}.$$

   Using the same identity, with $r = x^{10}$, we have

$$1 + x^{10} + x^{20} + x^{30} + ... + x^{90} = \frac{1 - x^{100}}{1 - x^{10}}.$$

   Similarly,

$$1 + x^{100} + x^{200} + x^{300} + ... + x^{900} = \frac{1 - x^{1000}}{1 - x^{100}},$$

$$1 + x^{1000} + x^{2000} + x^{3000} + ... + x^{9000} = \frac{1 - x^{10000}}{1 - x^{1000}},$$

   et cetera. Therefore,

$$(1 + x + x^2 + x^3 + ... + x^9)(1 + x^{10} + x^{20} + x^{30} + ... + x^{90}) = \frac{1 - x^{10}}{1 - x} \cdot \frac{1 - x^{100}}{1 - x^{10}} = \frac{1 - x^{100}}{1 - x},$$

$$(1 + x + ... + x^9)(1 + x^{10} + ... + x^{90})(1 + x^{100} + ... + x^{900}) = \frac{1 - x^{10}}{1 - x} \cdot \frac{1 - x^{100}}{1 - x^{10}} \cdot \frac{1 - x^{1000}}{1 - x^{100}} = \frac{1 - x^{1000}}{1 - x},$$

$$(1 + x + ... + x^9)(1 + x^{10} + ... + x^{90})(1 + x^{100} + ... + x^{900})(1 + x^{1000} + ... + x^{9000}) = \frac{1 - x^{10}}{1 - x} \cdot \frac{1 - x^{100}}{1 - x^{10}} \cdot \frac{1 - x^{1000}}{1 - x^{100}} \cdot \frac{1 - x^{10000}}{1 - x^{1000}} = \frac{1 - x^{10000}}{1 - x},$$

$$(1 + x + ... + x^9)(1 + x^{10} + ... + x^{90})(1 + x^{100} + ... + x^{900})(1 + x^{1000} + ... + x^{9000})(1 + x^{10000} + ... + x^{90000}) = \frac{1 - x^{10}}{1 - x} \cdot \frac{1 - x^{100}}{1 - x^{10}} \cdot \frac{1 - x^{1000}}{1 - x^{100}} \cdot \frac{1 - x^{10000}}{1 - x^{1000}} \cdot \frac{1 - x^{100000}}{1 - x^{10000}} = \frac{1 - x^{100000}}{1 - x},$$

   and so on. If $|x| < 1$, then the numerators of these results are approaching 1-0 = 1 as we carry out the product further and further. Therefore, the value of the *infinte* product must be $\frac{1-0}{1-x} = \frac{1}{1-x}$.