Eastern Shore High School Mathematics Competition
November 10, 2004
Team Contest

Answer as many questions as possible in the time provided. To receive full credit for a correct solution, show all work *and* provide a clearly written explanation. Solutions will be judged based on correctness, completeness and clarity. (Little credit, if any, will be given for a solution consisting of just a number or a single sentence.)

1. We have provided the pieces for the game pictured here. The object is to interchange the blue and the red pegs. The rules are:

   - You can move a peg to an adjacent hole.
   - You can jump an adjacent peg of the opposite color if there is an empty hole behind it.
   - You can't move a peg backwards.
   - You must start and finish with exactly one empty hole in the middle.

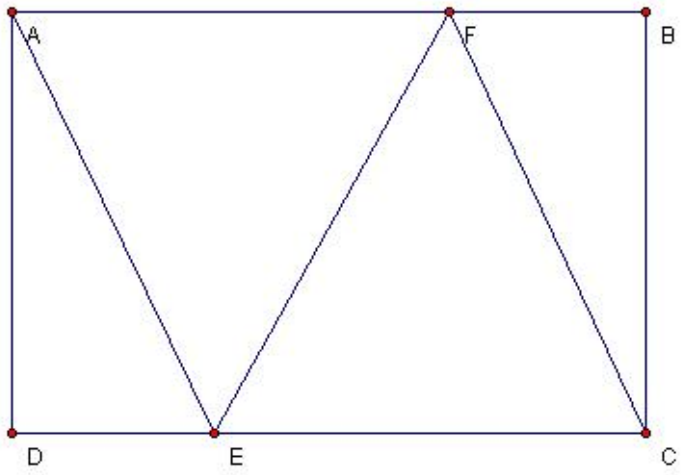
   (a) Complete the following table:

Number of pegs of each color:	1	2	3	4
Minimum number of moves needed to complete the game:				

   (b) Suppose we had a peg board long enough to hold x pegs of each color, with one hole left over in the middle, for any positive integer x. Find a formula to relate the number of pegs of each color (x) to the minimum number of moves it would require to complete the game. In your formula, use x for the number of pegs of each color, and y for the minimum required number of moves.

2. A seven man college household decided to hold a formal dinner party. Each member of the household was to bring a date and, to add an interesting twist, each of them was to bring flowers for one of the other member's dates. They arranged it so each of them would bring flowers for one date and each date would receive flowers from someone other than her escort. Barry gave orchids to the date of the man who gave flowers to David's date. Alan gave roses to Barry's date. Alan's date was delighted with the bouquet she received from the man whose date got Farrell's chrysanthemums. Earl's date received flowers from the man whose date received flowers from Carl. David bought flowers for the date of the man who gifted the date of the donor of Gary's date's flowers. Who gave flowers to Farrell's date?

3. (a) In the rectangle $ABCD$, suppose $AB = 16$ and $BC = 12$. The points E and F are chosen so that $AFCE$ is a rhombus. What is the length of EF?



(b) Repeat part (a), but this time suppose that $AB = a$ and $BC = b$, where $a > b$, and points E and F are chosen so that $AFCE$ is a rhombus. What is the length of EF in terms of a and b?

4. Let us call a number a "purple" number if (and only if) it satisfies both of the following conditions:

- It is a positive integer with two or more digits.
- The sum of the squares of its digits is equal to the two-digit number formed by its first two digits (counting from the leftmost decimal place).

For example, 2004 is a purple number, since it is a positive integer, its first two digits (counting from the left) are 20, and $2^2 + 0^2 + 0^2 + 4^2 = 20$.

Find each of the following numbers, and in each case explain how you know that your answer is the smallest (for parts a and b) or largest (for part c) possible.

(a) Find the smallest purple number that is greater than 2004.

(b) Find the smallest purple number.

(c) Find the largest purple number that is less than 10,000.

5. For $|x| < 1$, evaluate the infinite product:

$$(1 + x + x^2 + x^3 + ... + x^9)(1 + x^{10} + x^{20} + x^{30} + ... + x^{90})(1 + x^{100} + x^{200} + x^{300} + ... + x^{900}) \ldots .$$