Eastern Shore High School Mathematics Competition
November 10, 2004
Individual Contest

The correct response for each question is highlighted in **boldface** type.

1. Find the area of the region (in the xy plane) enclosed by the lines $y = 2x$, $x = 3$ and $y = 0$.

   (a) $\sqrt{18}$ (b) 6 (c) $\sqrt{45}$ (d) **9** (e) 18

   Solution: The lines $y = 2x$ and $x = 3$ intersect at (3,6), so the region described is a triangle with vertices (0,0), (0,3) and (6,3). Thus, is a right triangle with legs of lengths 3 and 6, and so its area is $A = \frac{1}{2}(3)(6) = 9$.

2. A group of students consists of 15 males and 25 females. Among them are 10 sophomores, 18 juniors, and 12 seniors. The sophomores are comprised of 6 females and 4 males. If one student is randomly selected to receive tickets to a concert, what is the probability that the selected student is a female sophomore?

   (a) **6/40** (b) 6/25 (c) 10/40 (d) 6/10 (e) 25/40

   Solution: Since we are randomly choosing one student from a group of 40 students, and 6 of these are female sophomore, the probability of choosing a female sophomore is:

$$p = \frac{\text{the number of female sophomores}}{\text{the total number of students}} = \frac{6}{40}.$$

3. What is the 2004^{th} term of the sequence $\{a + k, a + 2k, a + 3k, a + 4k, \ldots\}$ if the third term is 4 and the fourth term is 6?

   (a) 2004 (b) 2006 (c) 4008 (d) 4010 (e) **None of these**

   Solution: We're given that $a + 3k = 4$ and $a + 4k = 6$; it follows that $a = -2$ and $k = 2$, so that the k^{th} term of the sequence is $2k - 2$. Therefore, the 2004^{th} term of the sequence is 4006.

4. How many factors does 2004 have?

   (a) 2 (b) 6 (c) 9 (d) **12** (e) 16

   Solution: The factors of 2004 are: 1, 2, 3, 4, 6, 12, 167, 334, 501, 668, 1002 and 2004.

   Alternate solution: Aside from listing the factors, another way to determine the number of factors is to look at the prime factorization: $2004 = 2^2 \cdot 3^1 \cdot 167^1$. It follows that 2004 has $(2+1)\cdot(1+1)\cdot(1+1) = 3 \cdot 2 \cdot 2 = 12$ factors.

5. Let $f(x) = \log_{10} x$, and let $a = (10,000,000,000)^{1000}$. What is the value of $f(f(a))$?

   (a) **4** (b) 400 (c) $1000a$ (d) a^{1000} (e) None of these

   Solution:
   First, $f(a) = \log_{10}\left((10,000,000,000)^{1000}\right) = 1000\log_{10}(10,000,000,000) = 1000(10) = 10,000$. Therefore, $f(f(a)) = f(10,000) = \log_{10}(10,000) = 4$.

6. Suppose you are the payroll manager for a firm that employs 100 people. From data on the first week of July last year, the median wage was \$480, the mean wage was \$500, and the modal wage was \$475. Assuming positions and wages have remained stable, except that the overall wage level has increased 5%, how much money is needed to pay these 100 employees this year for the first week of July?

(a) \$47,500 (b) \$48,000 (c) \$50,000 (d) \$50,400 (e) $\mathbf{\$52,500}$

Solution: Since the *mean* wage was \$500 per worker last year for 100 workers, the total wages paid was \$50,000. If we increase the wage level by 5%, the new total is $(1.05)\$50,000 = \$52,500$.

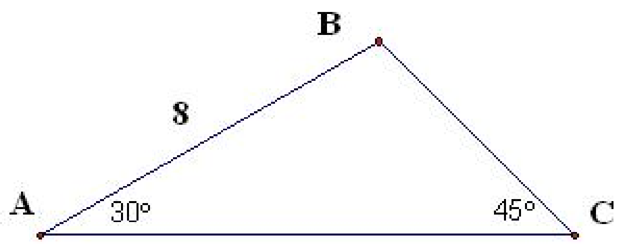
7. If $0^\circ < A < 90^\circ$ and $\tan A = \frac{2}{3}$, then what is the value of $\sin A$?

(a) $\frac{2}{5}$ (b) $\frac{1}{\sqrt{5}}$ (c) $\boldsymbol{\frac{2}{\sqrt{13}}}$ (d) $\frac{3}{\sqrt{13}}$ (e) $\frac{2}{\sqrt{5}}$

Solution: Since $0^\circ < A < 90^\circ$, we may use right triangle trigonometry: sketch a right triangle with legs of lengths 2 and 3. Then, the smaller of the two acute angles in this triangle is A, since we've constructed it in such a way that $\tan A = \frac{2}{3}$. Since the hypotenuse of this triangle is $\sqrt{2^2+3^2} = \sqrt{13}$, we have $\sin A = \frac{2}{\sqrt{13}}$.

Alternate solution: since $\tan A = \frac{\sin A}{\cos A}$, we know that $\frac{\sin A}{\cos A} = \frac{2}{3}$, and so $\cos A = \frac{3}{2}\sin A$. Since $\sin^2 A + \cos^2 A = 1$, we may use substitution to get the equation $\sin^2 A + \frac{9}{4}\sin^2 A = 1$. Solving this equation for $\sin^2 A$ gives us $\sin^2 A = \frac{4}{13}$, and so $\sin A = \frac{2}{\sqrt{13}}$.

8. In the triangle shown below, $m\angle CAB = 30^\circ$, $m\angle BCA = 45^\circ$, and side $\overline{AB}$ has length 8. What is the length of side $\overline{BC}$?



(a) 4 (b) $\frac{8}{\sqrt{3}}$ (c) $\mathbf{4\sqrt{2}}$ (d) $4\sqrt{3}$ (e) $5\sqrt{2}$

Solution: Draw an altitude from vertex B to side AC; use D to label the point at which this altitude intersects AC. Line segment BD divides $\triangle ABC$ into two right triangles: $\triangle ADB$, and $\triangle BDC$. Further, $\triangle ADB$ is a $30^\circ - 60^\circ - 90^\circ$ triangle, and $\triangle BDC$ is a $45^\circ - 45^\circ - 90^\circ$ triangle. Therefore, we may use the side ratios of these triangles to solve the problem: since $AB = 8$, $BD = \frac{1}{2} \cdot AB = \frac{1}{2}(8) = 4$, and since $BD = 4$, $BC = \sqrt{2} \cdot BD = 4\sqrt{2}$.

9. In the xy plane, what is the reflection of the point (-1,2) over the line $y = x$?

(a) (-2,1) (b) (-1,-2) (c) (1,-2) (d) (1,2) (e) $\mathbf{(2,-1)}$

Comment: for any point (x,y) in the xy plane, reflection over the line $y = x$ will reverse the x and y coordinates of the original point.

10. On planet Zelda, it is a well known fact that if one plays gibner, then one is skilled at nevil. Which of the following statements would follow logically from this?

    (a) Anyone skilled at nevil must play gibner
    (b) Those who do not play gibner are not skilled at nevil
    (c) Anyone unskilled at nevil must not play gibner
    (d) All of the above
    (e) None of the above

    Solution: Statements (a), (b) and (c) are the converse, inverse and contrapositive, respectively, of the given conditional statement. A conditional statement must be logically equivalent to its contrapositive, but not necessarily to its converse or inverse. Therefore, the only correct answer is (c).

11. A group of students completed a survey about their favorite campus sports to watch. The sports include football, soccer, and basketball. Students were allowed to choose more than one sport on the survey. The results were as follows: 30 students said they like to watch only football; 35 said they like to watch only soccer; 100 said they like to watch only basketball; 15 like to watch football and soccer; 15 like to watch football and basketball; 20 like to watch soccer and basketball; and 5 students answered that they like to watch all three sports. How many students completed the survey form?

    (a) 110 (b) 120 (c) 155 (d) 165 (e) **None of these**

    Solution: The actual answer to this question is 30+35+100+15+15+20+5=220. If you read the results carefully, you will see that all of the groups are disjoint, so that the inclusion-exclusion rule does not apply.

    For example: 30 students watch *only* football, 35 watch *only* soccer, and 15 watch *both* football and soccer. The total number of students described here is $30 + 35 + 15 = 80$. If the word *only* did not appear in these descriptions, then the 15 students who watch both would be the overlap between the two other groups; in this case, the answer would be 30+35-15=50.

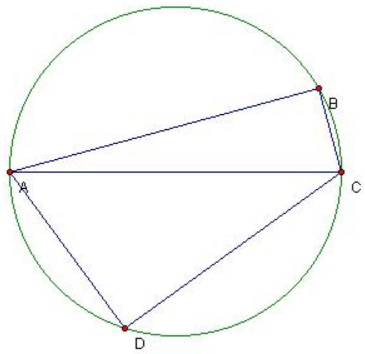
12. Solve the equation $\log_{\sqrt{2}} x - \log_{\sqrt{2}} 4 = 2$.

    (a) 0.5 (b) 2 (c) 6 (d) **8** (e) None of these

    Solution:

$$\begin{aligned}
\log_{\sqrt{2}} x - \log_{\sqrt{2}} 4 &= 2 \\
\log_{\sqrt{2}} \left(\frac{x}{4}\right) &= 2 \\
\frac{x}{4} &= \left(\sqrt{2}\right)^2 = 2 \\
x &= 8
\end{aligned}$$

13. Given that AC is a diameter of the circle shown below, what is the sum of the measures of $\angle BAD$ and $\angle BCD$?



(a) 90° (b) **180°** (c) 200° (d) 270°
(e) Cannot be determined from the given information

Solution: Since each of $\triangle\ ABC$ and $\triangle\ ADC$ both have diameter AC as its longest side, it follows that each is a right triangle. It follows that

$$m\angle BAC + \angle BCA = 90°, \text{ and } m\angle DAC + \angle DCA = 90°.$$

Therefore,

$$\begin{aligned} m\angle BAD + m\angle BCD &= (m\angle BAC + m\angle DAC) + (m\angle BCA + m\angle DCA) \\ &= (m\angle BAC + m\angle BCA) + (m\angle DAC + m\angle DCA) \\ &= 90° + 90° = 180°. \end{aligned}$$

Comment: The assumption that AC is a diameter of the circle is in fact unnecessary; the result would hold for *any* chord AC. This is because the measure of an inscribed angle of a circle is equal to half the measure of its intercepted arc. Since the measures of arcs BAD (the intercepted arc of $\angle BCD$) and BCD (the intercepted arc of $\angle BAD$) add up to 360°, it follows that $m\angle BAD + \angle BCD = 180°$.

14. Points K and L move with a constant rotational velocity in a counterclockwise direction around a circle with center at the origin and a radius of 3. If the rotational velocity of K is $\frac{5\pi}{6}$ and that of L is $\frac{\pi}{3}$, and if both are located at (3,0) at time $t = 0$, how may units of time will pass before K and L again coincide?

(a) 3 (b) 3.5 (c) **4** (d) 4.5 (e) 5

Solution: The rotational velocity of each particle is defined as the distance travelled per unit of time divided by the radius, or (more simply) as the number of radians traversed per unit of time. Since there are 2π radians in a circle (regardless of its radius), the two particles will coincide whenever the difference between their distances travelled (measured in radians) is a multiple of 2π. So, we must find the least positive solution of the equation

$$\frac{5\pi}{6}t - \frac{\pi}{3}t = 2\pi n, \quad \text{where } n \text{ is an integer:}$$

$$\begin{aligned}
\frac{5\pi}{6}t - \frac{\pi}{3}t &= 2\pi n \\
6\left(\frac{5\pi}{6}t - \frac{\pi}{3}t\right) &= 6 \cdot 2\pi n \\
5\pi t - 2\pi t &= 12\pi n \\
3\pi t &= 12\pi n \\
t &= 4n
\end{aligned}$$

It follows that the *least* positive solution for t occurs when $n = 1$; that is, $t = 4$. Therefore, the particles again coincide after 4 units of time.

15. How many real numbers k are there such that the equations $x^2 - kx + 8 = 0$ and $x^2 + k = 0$ each have two integer solutions x?

(a) 0 (b) **1** (c) 2 (d) 3 (e) More than 3

Solution: The first equation has two integer solutions a, b if and only if $x^2 - kx + 8$ can be factored as $(x - a)(x - b)$, which requires that $ab = 8$ and $a + b = k$. The only such possibilities are the ordered pairs (a,b)=(8,1), (4,2), (-8,-1) and (-4,-2), which correspond to k=9,6,-9 and -6, respectively. The only one of these possibilities for which the second equation has integer solutions is $k = 9$; in this case, we have $x^2 - 9 = 0$, which has integer solutions 3 and -3. Thus, the only real number for which both equations have two integer solutions is $k = -9$.

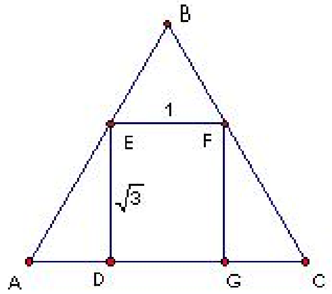
16. If A, B and C are sets, then the statement $x \in A \cap (B \cup C)$ is logically equivalent to which of the following?

(a) x is in A and x is in B or C.
(b) x is in A or x is in B and C.
(c) x is in A and B but not C or x is in A and C but not B.
(d) x is in at least one of the three sets.
(e) x is in all three sets.

Solution: We interpret $\cap$ as "and"; that is, x to be in the intersection of two sets if and only if x is in one set *and* (also) in the other set. We then interpret $\cup$ as "or": x is in the union of B and C if and only if x is in B or x is in C.

Comment: In mathematics, the word "or" is properly interpreted in the *inclusive* sense. For example, in everyday language, the final phrase of the preceding paragraph might be stated, in everyday (non-mathematical) language as "x is in B and/or x is in C", or "x is in B or C or both."

17. In the diagram below, $\triangle ABC$ is an equilateral triangle, DEFG is a rectangle, $DE = \sqrt{3}$ and $EF = 1$. What is the perimeter of $\triangle ABC$?



(a) $3(\sqrt{3}+1)$ (b) **9** (c) $3(2\sqrt{3}+1)$ (d) 15 (e) 21

Solution: Since $\triangle ABC$ is equilateral, we may simply find length AB and triple it. Note that $\triangle BEF$ is equilateral (all three angles have measure $60°$); therefore, $BE = EF = 1$. Also, note that $\triangle ADE$ is a $30° - 60° - 90°$ triangle; therefore, we can use the side ratios of such a triangle to conclude that $AE = 2$. It follows that $AB = AE + EF = 2 + 1 = 3$, and thus the perimeter of the triangle is $3 \times 3 = 9$.

18. A ball is dropped from 10 meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance of $\frac{3}{5}h$. Find the total distance the ball travels up and down if it is allowed to continue bouncing indefinitely.

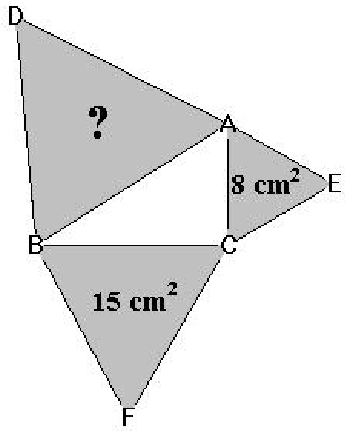
(a) 40 meters (b) $1000\sqrt{3}$ meters (c) $1000\sqrt{5}$ meters
(d) An infinite distance (e) None of these

Solution: Let's look at the first few bounces. After the initial 10m drop, the ball bounces up $\frac{3}{5} \cdot 10m = 6m$. The ball then drops 6m, and then it bounces up $\frac{3}{5} \cdot 6m = \frac{3}{5}\left(\frac{3}{5} \cdot 10m\right) = 3.6m$. And so on; since the height of each bounce is $\frac{3}{5}$ the height of the preceding bounce, we find that the height of the n^{th} bounce will be $10 \cdot (3/5)^n$ for all n. Thus, we may represent the total distance travelled up and down by the ball as follows:

$$\begin{aligned}
\text{Total distance} &= 10 + 6 + 6 + 3.6 + 3.6 + \ldots + 10 \cdot \left(\frac{3}{5}\right)^n + 10 \cdot \left(\frac{3}{5}\right)^n + \ldots \\
&= 10 + 2\left(6 + 3.6 + \ldots + 10 \cdot \left(\frac{3}{5}\right)^n + \ldots\right), \quad \text{or} \\
&= 10 + 2\sum_{n=1}^{\infty} 6 \cdot \left(\frac{3}{5}\right)^{n-1} \qquad \text{(written in sigma notation).}
\end{aligned}$$

The general formula for the sum of an infinite geometric series, $\sum_{n=1}^{\infty} ar^{n-1}, |r| < 1$ is $\frac{a}{1-r}$. Therefore, the value of the above sum is $10 + 2 \cdot \frac{6}{1-3/5} = 10 + \frac{12}{2/5} = 10 + 30 = 40$.

19. Given that $\triangle\ ABC$ is a right triangle, triangles $\triangle\ ABD$, $\triangle\ ACE$ and $\triangle\ BCF$ are all equilateral and the areas of triangles $\triangle\ ACE$ and $\triangle\ BCF$ are 8 cm^2 and 15 cm^2 respectively, what is the area of $\triangle\ ABD$?



(a) $\sqrt{120}$ cm^2 (b) 17 cm^2 (c) 20 cm^2 (d) **23 cm^2** (e) None of these

Solution #1: One approach to this problem is to find the side length of each triangle. The height of an equilateral triangle of side length s is $h = \frac{s\sqrt{3}}{2}$; it follows that the area of such a triangle is $A = \frac{1}{2} \cdot s \cdot \frac{s\sqrt{3}}{2} = \frac{s^2\sqrt{3}}{4}$. we can solve for s to find that the side length of an equilateral triangle with area A is $s = \sqrt{\frac{4A}{\sqrt{3}}}$. It follows that $AC = \sqrt{\frac{32}{\sqrt{3}}}$ and $BC = \sqrt{\frac{60}{\sqrt{3}}}$. By the Pythagorean Theorem, we find: $AB^2 = AC^2 + BC^2 = \frac{92}{\sqrt{3}}$, and therefore the area of $\triangle\ ABD$ is: $\frac{AB^2\sqrt{3}}{4} = \frac{(92/\sqrt{3})\sqrt{3}}{4} = \frac{92}{4} = 23$.

Solution # 2: Instead of the approach described above, one may use proportional thinking to find the solution with much less calculation. In particular, observe that all equilateral triangles are geometrically similar. This is useful because the areas of similar figures are proportional to the *squares* of the corresponding side lengths. Therefore, there exists a constant k such that, for all such triangles, $s^2 = kA$, where $A=$ the area of the figure with side length s. It follows that, whatever the value of k happens to be, $AC^2 = 8k$ and $BC^2 = 15k$. By the Pythagorean Theorem, $AB^2 = AC^2 + BC^2 = 23k$, and so the area of $\triangle\ ABD = 23$.

(Note that by this approach, we have no need to know that for equilateral triangles, k happens to be $\sqrt{3}/4$. Simply knowning that the constant holds for all equilateral triangles is enough.)

20. Suppose a city legislature consists of 12 Whigs and 8 Tories. From this group, the city's mayor wishes to select a committee consisting of 3 Whigs and 2 Tories. In how many different ways could the mayor select this committee?

(a) $\mathbf{\dfrac{12!\,8!}{9!\,6!\,3!\,2!}}$ (b) $\dfrac{15!}{12!\,8!\,5!}$ (c) $\dfrac{20!\,3!\,2!}{12!\,8!\,5!}$ (d) $\dfrac{20!\,5!}{10!\,5!\,3!\,2!}$ (e) $\dfrac{25!}{12!\,8!\,3!\,2!}$

Solution: To solve this problem, one must use combinations. We will use here the notation $\binom{n}{k}$ to denote the number of combinations of k elements from n. (In other words, this is the number of subsets of k elements from a set of n elements.) The formula for this quantity is $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

There are $\binom{12}{3}$ ways to choose 3 Whigs from 12, and $\binom{8}{2}$ ways to choose 2 Tories from 8; therefore, there is a total of

$$\binom{12}{3}\binom{8}{2} = \frac{12!}{3!\,9!}\cdot\frac{8!}{2!\,6!} = \frac{12!\,8!}{9!\,6!\,3!\,2!}$$

ways in which the mayor could choose the committee.