Eastern Shore High School Mathematics Competition
November 13, 2002
Team Round Solutions

1. The students in a geometry class are investigating the question of whether three given line segments will form a triangle. They know that a triangle is formed if and only if the sum of the lengths of any two segments is greater than the length of the third segment.

   The students have two wooden rods, one of which is longer than the other. They cut the longer rod at a randomly chosen spot, so that they now have three rods. What is the probability that the three lengths form a triangle...

   (a) If the longer rod (before being cut) was twice as long as the shorter rod?

   (b) If the longer rod was three times as long as the shorter rod?

   (c) If the longer rod was N times as long as the shorter rod?

   Solutions: For each question, we must break the longer rod so that the *difference* between the lengths of the two pieces is greater than the length of the shorter rod.

   We start with part (a). Let the lengths of the rods be given by n and $2n$. The longer rod will be cut into two pieces, of lengths k and $2n - k$, where k is a number chosen at random from the interval $(0, n]$. (Here we are using "k" to denote the length of the *shorter* of the two pieces). We will be able to form a triangle if, and only if, $(2n - k) - k < n$. Solve for k, and we find that we must have $k > \frac{n}{2}$. Since $k \leq n$, this means $\frac{n}{2} < k \leq n$. The probability of choosing k from this interval (of length $\frac{n}{2}$) from the interval $(0, n]$ (which is of length n) is $\dfrac{n/2}{n} = \dfrac{1}{2}$.

   For part (b), proceed similarly: we must choose k, $0 < k \leq \frac{3n}{2}$, such that $(3n - k) - k < n$. It follows that we must have $n < k \leq \frac{3n}{2}$, and the probability of such a random choice is $\dfrac{n/2}{3n/2} = \dfrac{1}{3}$.

   In general, we end up requiring that k, being randomly chosen from $0 < k \leq \frac{Nn}{2}$, must fall within the interval $\dfrac{(N-1)n}{2} < k \leq \dfrac{Nn}{2}$; the probability of such a choice is $\dfrac{n/2}{Nn/2} = \dfrac{1}{N}$.

2. The sum of the ages of Mike and Brenda's children is 17. (Here we consider each child's age as a *whole number*.) Given this information, what is the largest possible product of the childrens' ages...

   (a) if Mike and Brenda have exactly two children?

   (b) if Mike and Brenda have more than two children?

   Solutions:

   (a) It's easy to determine (through checking all possible cases) that the product is maximized when the ages are 8 and 9, giving a product of 72.

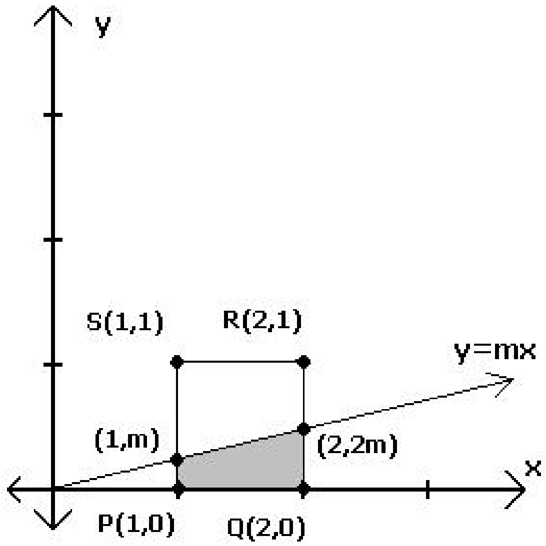
   (b) If more than two children are allowed, one must consider all possible "partitions" of 17, to find the highest possible product.

   It turns out that the product is maximized when there are as many 3-year old children as possible. (This would be true for any age sum, not just for 17.) So, to maximize the product of the childrens' ages, we'd have children of ages 3,3,3,3,3 and 2. (Three-year old quintuplets!) The resulting product is 486, which is the maximum possible.

   Further comments on #2:
   While it's not easy to rigorously *prove* that the strategy of maximizing the number of 3's is always the best for this type of problem, we can partially justify it as follows: suppose (for example) there is a child of age 5. We can replace that child with two children of ages 3 and 2; this would keep the overall age sum constant, but increase the product (since $3 \cdot 2 = 6 > 5$. Similarly, two three-year olds are "preferable" to three two-year olds, since $2 + 2 + 2 = 3 + 3$ (leaving the sum unchanged) but $2 \cdot 2 \cdot 2 < 3 \cdot 3$...

3. A square is drawn on a (rectangular) coordinate plane with vertices P(1,0), Q(2,0), R(2,1) and S(1,1). A line is drawn from the origin that cuts the square into two sections, the ratio of whose areas is 2:1. What's the equation of the "cutting" line?

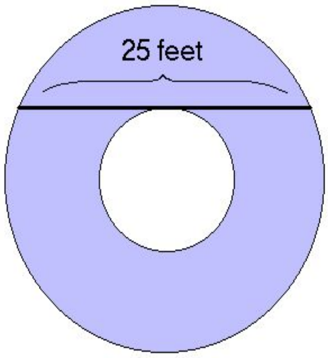
Solution: A diagram is usually helpful for geometric problems.



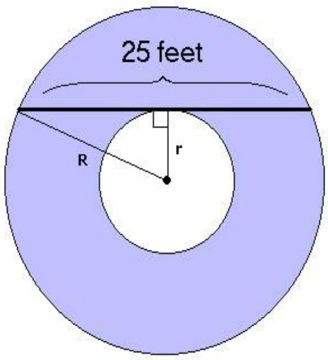
The equation of a line through the origin has an equation of the form $y = mx$, where m is a constant. Note that, if it crosses the two vertical sides of the square, then it will intersect them at (1,m) and (2,2m). The area of the shaded trapezoid, then, will be $\frac{3}{2}m$. To form the required ratio, the area of this trapezoid must be $\frac{1}{3}$ (if it is the smaller piece) or $\frac{2}{3}$ (if it is the larger piece). Solving for m in each case, we get $m = \frac{2}{9}$ and $m = \frac{4}{9}$. Therefore, there are in fact *two* correct answers: $y = \frac{2}{9}x$, and $y = \frac{4}{9}x$.

Additional comment: This question was, admittedly, slightly tricky in that there are two possible answers, and this possibility is not explicitly indicated by the instructions. However, it's always important to avoid making unwarranted assumptions! For example, it is never stated that the 2:1 ratio indicates the ratio of the top piece to the bottom piece; it could be the reverse! Therefore, both possibilities must be accounted for, and as a result there are *two* correct answers for this problem.

4. A father and son were hired to paint the floor of a merry-go-round. The floor had the shape of a washer (see the diagram below). Because the machinery to drive the merry-go-round was located in the center of the concentric circles, the son stretched a 25' measuring tape across the interior of the outer circle, but avoided the center; it lay, by chance, so that it just touched the edge of the inner circle (as shown in the diagram.) To his father's surprise, the son was able to determine the area to be painted from this measurement alone. What was the area to be painted?



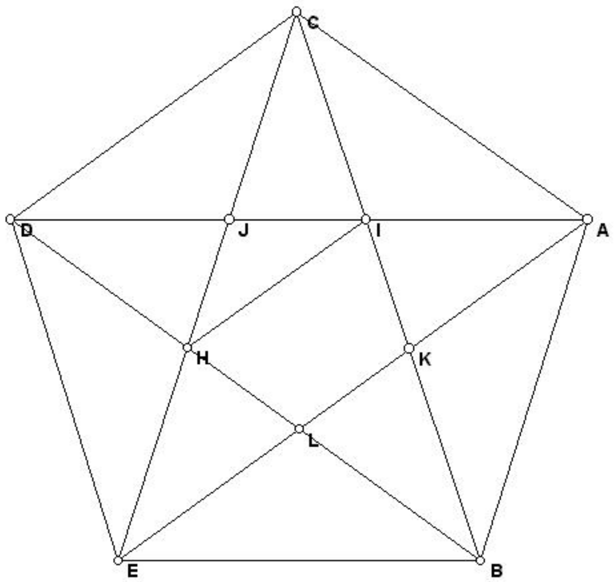
Solution: Consider the modified version of the above diagram:



Here we use r and R for the radii of the smaller circle and the larger circle, respectively. We can see that the shaded area will be the area of the larger circle minus the area of the smaller circle: $\pi R^2 - \pi r^2$, or $\pi(R^2 - r^2)$. We can also see, from the right triangle formed by the specific radii drawn in the diagram, that $r^2 + (25/2)^2 = R^2$; that is, $R^2 - r^2 = \dfrac{625}{4}$. Therefore, by substitution, we have a shaded area of $\frac{625}{4}\pi$, or 156.25π.

Additional comment: It is of interest that we do not need to (and, in fact, *can't*) solve for R and/or r; all we can deduce is the quantity $R^2 - r^2$, which in turn is the only quantity we need to solve the problem. The danger here is in trying to do *too much* (i.e., solving for both radii) instead of focusing precisely on what is *necessary* to solve the problem...

5. In the following regular pentagon with sides of length 1 (for example, $AB = 1$), what is the value of $\frac{HI}{HJ}$, where AB, HI and HJ are the lengths of the line segments $\overline{AB}$, $\overline{HI}$ and $\overline{HJ}$ respectively?



Here we will outline one method of solution for this problem. If one notices that HJIKL is a regular pentagon, just like ABEDC, then it follows (since any two regular pentagons must be geometrically similar) that the ratio HI/HJ will be equal to the ratio AE/AB. So, we can reduce the problem to simply finding the length of the line segment $\overline{AE}$ (assuming AB=1).

It is not difficult to show that triangle BAL is isosceles; therefore, AL=AB=1. Similarly, EK=1; therefore, $AE = AL + KE - KL = 2 - KL$. Note also that $AK = AL - KL = 1 - KL$.

Next, we make use of the fact (also not difficult to establish) that triangles ABE and AKB are similar, since both have two 36° angles. Therefore, $\frac{AE}{AB} = \frac{AB}{AK}$. By substitution, this gives us the equation $\frac{2 - KL}{1} = \frac{1}{1 - KL}$. Now, since $AE = 2 - KL, KL = 2 - AE$, so by substitution we now have $\frac{AE}{1} = \frac{1}{AE - 1}$. If we solve for AE, we find that $AE = \frac{1 \pm \sqrt{5}}{2}$; since AE is larger than 1, the answer is $\frac{1 + \sqrt{5}}{2}$.

Note: this answer is also known as the "Golden Ratio."