

Eastern Shore High School Mathematics Competition
November 13, 2002
Individual Round Solutions

1. An author published a book every two years. When the seventh one was published, the sum of the years of publication was 13,804. In what year was the author's first book published?
(a) 1960 (b) 1964 (c) 1966 (d) 1970 (e) 2002

Solution: The correct answer is **c**. To see this, let x denote the year of the first book's publication. Then, the six successive books were published in the years $x + 2$, $x + 4$, $x + 6$, $x + 8$, $x + 10$ and $x + 12$. The sum of the seven years, then, is $7x + 42$. If we set $7x + 42$ equal to 13,804 and solve for x , the solution is $x = 1,966$.

2. Suppose x and y are positive numbers such that $\log_2(x + y) = \log_2 y + 10$. Which of the following is true?
(a) $x = 1023y$ (b) $\log_2 x = 10$ (c) $x = 2^{10}$ (d) $x + y = 2^y + 10$ (e) None of these

Solution: The correct answer is **a**:

$$\begin{aligned}\log_2(x + y) &= \log_2 y + 10 \\ \log_2(x + y) &= \log_2 y + \log_2(2^{10}) \\ \log_2(x + y) &= \log_2 y + \log_2 1024 \\ \log_2(x + y) &= \log_2(1024y) \\ x + y &= 1024y \\ x &= 1023y\end{aligned}$$

3. A second number is added to a first number, and it is observed that the sum of the two numbers is equal to their product. The second number is then multiplied by one less than the first number. What can we conclude about this new product?
(a) The product cannot be an integer.
(b) The product must be smaller than the second number.
(c) The product cannot be negative.
(d) The product must be equal to the first number.
(e) The product cannot be zero.

Solution: The correct answer is **d**. To see this, call the first number x and the second number y . Then, the first two sentences above become:

- (a) $x + y = xy$
(b) What can we conclude about $(x - 1)y$?

The answer: since $(x - 1)y = xy - y$, and we've already been told that $xy = x + y$, we may substitute $x + y$ for xy to obtain $(x - 1)y = (x + y) - y = x$. Therefore, the product $(x - 1)y$ is equal to the "first number," x .

4. Bob's neighbor, Mr. Smith, is taking a month's vacation in June. Bob will take care of Mr. Smith's house and lawn for all of the 30 days Mr. Smith will be away. Mr. Smith has developed an unusual pay scheme to reward Bob for his month's work: Bob will receive \$10 on June 1st, and thereafter his pay for any given day will be three dollars more than it had been for the day before. How much money will Bob receive for all 30 days?
(a) \$100 (b) \$1,605 (c) \$3,210 (d) \$10,000 (e) Bob will earn millions of dollars.

Solution: The correct answer is **b**.

Note that Bob will earn $\$10 + \$13 + \dots + \$97$ in all. (The last day's sum is $\$10 + 3 \cdot \29 .) To find this sum, one may spend a few minutes punching buttons on a calculator; alternatively, one may find a more clever approach. For example, notice:

- Bob's total for Day 1 and Day 30 is $\$10 + \$97 = \$107$.
- Bob's total for Day 2 and Day 29 is $\$13 + \$94 = \$107$.
- Bob's total for Day 3 and Day 28 is $\$16 + \$91 = \$107$.
- And so on...

Proceeding in this way – and noting that there are fifteen pairs of days in June – one may conclude that Bob will earn $15 \times \$107 = \$1,605$ in all.

5. Which of the following expressions is equivalent to $(x + y)^{-1}(x^{-1} + y^{-1})$?
- (a) $x^{-2} + 2x^{-1}y^{-1} + y^{-2}$
(b) $x^{-2} + 2^{-1}x^{-1}y^{-1} + y^{-2}$
(c) $\frac{1}{x^{-1}y^{-1}}$
(d) $x^{-2} + y^{-2}$
(e) $x^{-1}y^{-1}$

Solution: The correct answer is **e**:

$$\begin{aligned}(x + y)^{-1}(x^{-1} + y^{-1}) &= \frac{\frac{1}{x} + \frac{1}{y}}{x + y} \\ &= \frac{\left(\frac{1}{x} + \frac{1}{y}\right)(xy)}{(x + y)(xy)} \\ &= \frac{y + x}{(x + y)xy} \\ &= \frac{1}{xy} = x^{-1}y^{-1}\end{aligned}$$

6. Suppose x , y and z are positive integers such that $x^2 + y^2 = z^2$. Which of the following must be true?

- I. $(x + y + z)^2 = 2(z + x)(z + y)$
- II. $(x - y - z)^2 = 2(z - x)(z - y)$
- III. $(x - y + z)^2 = 2(z + x)(z - y)$

(a) I only (b) I & III only (c) II & III only (d) II only (e) I, II & III

Solution: The correct answer is **b**.

As an example, we will show why equation I above is true. (Equation III may be verified, and II rejected, in much the same way.) We begin by expanding the left-hand side of equation I:

$$\begin{aligned}(x + y + z)^2 &= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz \\ &= 2z^2 + 2xy + 2xz + 2yz \quad (\text{since } z^2 = x^2 + y^2) \\ &= 2(z^2 + zy + xz + xy) \\ &= 2(z(z + y) + x(z + y)) \\ &= 2(z + x)(z + y)\end{aligned}$$

7. Suppose A and B are events from the same sample space and $P(A) = \frac{3}{5}$. Under what circumstances, if any, can we conclude that the probability that A or B (or both) occurs is $\frac{3}{7}$?

- (a) A and B are mutually exclusive, and $P(B) = \frac{6}{35}$.
- (b) A and B are independent, and $P(B) = \frac{6}{35}$.
- (c) A and B are mutually exclusive, and $P(B) = \frac{9}{35}$.
- (d) A and B are independent, and $P(B) = \frac{36}{35}$.

(e) Under *no* circumstances.

Solution: The correct answer is **e**. In general, the probability of A and/or B is at least as high as the probability of A . Therefore, if $P(A) = \frac{3}{5}$, then the probability of A and/or B (which may also be written as $P(A \cup B)$) must be *at least* $\frac{3}{5}$.

8. How many positive integers are factors of 2002?

- (a) 2 (b) 4 (c) 8 (d) 16 (e) 32

Solution: The correct answer is **d**.

The key here is to note that the prime factorization of 2002 is $2 \cdot 7 \cdot 11 \cdot 13$. (In particular: 1001 is *not* prime.) Therefore, any positive integer whose prime factorization includes at most one each of 2, 7, 11 and 13, and no other prime factors, is a factor of 2002. There are sixteen such integers.

9. How many prime numbers are there between 1 and 100?

- (a) 9 (b) 16 (c) 25 (d) 49 (e) 50

Solution: The correct answer is **c**. They are:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

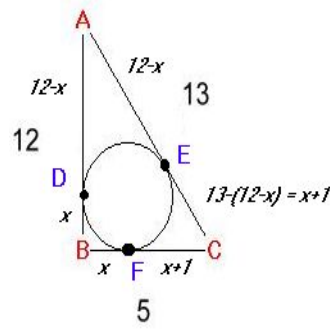
10. A circle is inscribed inside a 5-12-13 triangle, as shown below. What is the area of the circle?

- (a) 2π (b) 10 (c) 4π (d) 5π (e) 20

Solution: The correct answer is **c**.

From each vertex of the triangle, two tangents of equal length are drawn to the circle. The distance For example (see the below diagram), BD must equal BF . We let $x = BD$, and proceed “around” the triangle, to determine CD in terms of x . Since $BF + FC = 5$, we may set up an equation for x : $x + (x + 1) = 5$, and so $x = 2$. Finally, we note that x is also the radius

of the inscribed circle, so the area of the circle is $\pi x^2 = 4\pi$.



11. Which of the following is the largest?

- (a) $2^{10,000,000,000}$
- (b) $3^{6,000,000,000}$
- (c) $5^{4,000,000,000}$
- (d) $6^{3,500,000,000}$
- (e) $10^{1,000,000,000}$

Solution: The correct answer is **a**.

For this problem, we need only compare the *one-billionth roots* of these numbers, which are: 2^{10} , 3^6 , 5^4 , $6^{3.5}$, and 10^1 . When these numbers are computed, we find that 2^{10} is the largest; therefore, $2^{10,000,000,000}$ is the largest of the original five numbers.

12. If $(f \circ g)(x) = x^4$ and $f(x) = (x + 1)^2$, then a correct choice for $g(x)$ is:

- (a) $g(x) = x^2 - 1$
- (b) $g(x) = x^2 + 1$
- (c) $g(x) = x^4 + 1$
- (d) $g(x) = (x + 1)^4$
- (e) $g(x) = \frac{x^4}{(x + 1)^2}$

Solution: The correct answer is **a**.

The correct choice is $g(x) = x^2 - 1$, since we would have

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x^2 - 1) \\ &= ((x^2 - 1) + 1)^2 \\ &= (x^2)^2 = x^4, \text{ as required.}\end{aligned}$$

13. On the xy -plane, for fixed a , the symmetric axis of the curve $2x^2 + y - 4ax - \frac{3}{2} = 0$ is
 (a) $x = a$ (b) $x = -a$ (c) $x = -2a$ (d) $x = 2a$ (e) None of these

Solution: The correct answer is **a**.

To see this, put the equation into the standard form for a parabola, $y = a(x - h)^2 + k$:

$$\begin{aligned} y &= -2x^2 + 4ax + \frac{3}{2} \\ &= -2(x^2 - 2ax) + \frac{3}{2} \text{ complete the square for } x \\ &= -2(x^2 - 2ax + a^2 - a^2) + \frac{3}{2} \\ &= -2(x^2 - 2ax + a^2) + 2a^2 + \frac{3}{2} \\ &= -2(x - a)^2 + 2a^2 + \frac{3}{2} \end{aligned}$$

Thus, the parabola has vertex $(a, 2a^2 + \frac{3}{2})$, and so its axis of symmetry is the vertical line $x = a$.

14. What is the coefficient of x^4y^6 in the expansion of $(2x + 3y)^{10}$?
 (a) $\frac{10!}{4! \cdot 6!}$ (b) $2^4 \cdot 3^6 \cdot \frac{10!}{4! \cdot 6!}$ (c) $\frac{4! \cdot 6!}{2! \cdot 3!}$ (d) $2^6 \cdot 3^4 \cdot \frac{10!}{4! \cdot 6!}$ (e) $2^4 \cdot 3^6 \cdot \frac{4! \cdot 6!}{10!}$

Solution: The correct answer is **b**.

The Binomial Theorem states that the expansion of a binomial power, $(a + b)^n$, is a sum of terms of the form ${}_nC_k a^k b^{n-k}$, where ${}_nC_k = \frac{n!}{k! \cdot (n - k)!}$. (The quantities ${}_nC_k$ are called “binomial coefficients.”)

Applying the Binomial Theorem to this question, then, we see that the expansion of $(2x + 3y)^{10}$ is a sum of terms of the form $(2x)^k(3y)^{10-k} \frac{10!}{k! \cdot (10 - k)!}$. We’re interested in the term containing x^4y^6 , which corresponds to $k = 4$:

$$(2x)^4(3y)^6 \frac{10!}{4! \cdot 6!}, \text{ which is equal to } 2^4 3^6 \frac{10!}{4! \cdot 6!} x^4 y^6.$$

15. If A and B are sets such that $(A \cup B) \subseteq (A \cap B)$, then which of the following, if any, must be true?

- (a) $A = B'$ (b) $A = \emptyset$ (c) $B = \emptyset$ (d) $A = B$ (e) None of these

Solution: The correct answer is **d**.

The condition $(A \cup B) \subseteq (A \cap B)$ may be restated as follows: every element that is in either A or B (or both) must be contained in *both* A and B . In other words, each element of A must be contained in $A \cap B$, and each element of B must be contained in $A \cap B$. This is only possible if A and B are the same set.

16. The arithmetic mean (average) of the heights of the players on this year's basketball team was 72 inches. However, the tallest player, who is 75 inches tall, had to leave the team due to an injury. As a result, the arithmetic mean of the players' heights is now 71.8 inches. How many players were originally on the team?

- (a) 10 (b) 15 (c) 16 (d) 20 (e) Cannot be determined

Solution: The correct answer is **c**.

Let N equal the number of players originally on the team, and let H equal the sum of the heights of all of the players originally on the team. Then, we have the following information:

$$\begin{aligned}\frac{H}{N} &= 72, \text{ and} \\ \frac{H - 75}{N - 1} &= 71.8.\end{aligned}$$

If we solve both equations for H , we obtain:

$$\begin{aligned}H &= 72N, \text{ and} \\ H &= 71.8(N - 1) + 75 = 71.8N + 3.2.\end{aligned}$$

Therefore, $72N = 71.8N + 3.2$, and so $N = 16$.

17. The graphs of $y = -|x - a| + b$ and $y = |x - c| + d$ intersect at the points $(2, 5)$ and $(8, 3)$. What is the value of $a + c$?

- (a) 7 (b) 8 (c) 9 (d) 10 (e) 12

Solution: The correct answer is **d**.

First, substitute $x = 2, y = 5$ and $x = 8, y = 3$ into the first equation to obtain:

$$\begin{aligned} 5 &= -|2 - a| + b \\ 3 &= -|8 - a| + b, \end{aligned}$$

and subtract to get the equation $|8 - a| - |2 - a| = 2$. The only solution to this equation is $a = 4$. Similarly, we can substitute both ordered pairs into the second equation to find $|2 - c| - |8 - c| = 2$, whose only solution is $c = 6$. Therefore, $a + c = 4 + 6 = 10$.

18. Assume each of the following premises is true:

- All tennis players wear funny clothes.
- No one who wears funny clothes subscribes to “Southern Living.”
- Some golfers subscribe to “Southern Living.”
- John plays tennis.

Based on these premises, which of the following *must* be true?

- (a) John does not play golf.
(b) John does not subscribe to “Southern Living.”
(c) Some golfers play tennis.
(d) Some tennis players play golf.
(e) No one who wears funny clothes plays golf.

Solution: The correct answer is **b**.

John plays tennis (fourth premise); therefore, John wears funny clothes (by the first premise). It follows (by the second premise) that John does not subscribe to “Southern Living.”

19. Increase a rectangle's width by 1 and decrease its length by 1, and its area is decreased by 4. But decrease its length by 1 and increase its width by 2, and its area is increased by 16. What is the area of the rectangle?

(a) 16 (b) 45 (c) 88 (d) 180 (e) 504

Solution: The correct answer is **e** .

Let L and W be the length and width, respectively, of the rectangle. Then, the rectangle's area is LW . We're given $(L - 1)(W + 1) = LW - 4$, and $(L - 1)(W + 2) = LW + 16$. These equations may be rewritten as $L - W = -3$ and $2L - W = 18$. The solution to this system of equations is $L = 21$, $W = 24$, and so the rectangle's area is $21 \cdot 24 = 504$.

20. Suppose that there were thirty students in a class, twenty of whom were girls. Two students were selected from the class at random. You are told that at least one of the selected students was a girl. Given this information, what is the probability that *both* of the selected students were girls?

(a) 1/40 (b) 1/4 (c) 1/3 (d) 19/39 (e) 1/2

Solution: The correct answer is **d**.

(Note: here we will use the notation ${}_n C_k$ to denote the number of possible combinations of k elements chosen from n . The formula is ${}_n C_k = \frac{n!}{k! \cdot (n - k)!}$.)

There are ${}_{30} C_2$ possible pairs of students, each of which is equally likely to be selected. Of these, there are ${}_{20} C_2$ possible pairs of girls, and ${}_{10} C_2$ possible pairs of boys. We're given that the pair consists of at least one girl; there are ${}_{30} C_2 - {}_{10} C_2$ such possible pairs. (This is the total number of possible pairs, minus the number of pairs consisting of two boys.) Therefore, the probability that both students are girls, given that at least one is a girl, is

$$\frac{\# \text{ of pairs of two girls}}{\# \text{ of pairs with at least one girl}} = \frac{{}_{20} C_2}{{}_{30} C_2 - {}_{10} C_2} = \frac{190}{435 - 45} = \frac{190}{390} = \frac{19}{39}.$$