

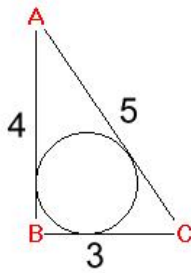
Eastern Shore Mathematics Competition - November 14, 2001
Team Problems with Partial Solutions

(Note: Solutions are provided at the end of this document.)

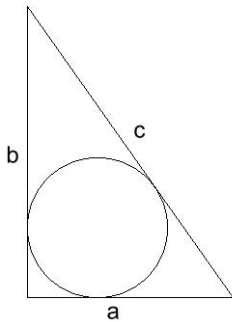
Instructions: Complete as many problems as you can in the time provided. To receive full credit for a problem, show all work and provide a clearly written explanation. (Little credit will be given for an answer consisting of just a number or a single phrase.)

1. For each of the following, find the area of the circle inscribed in the given right triangle.

(a)



(b)

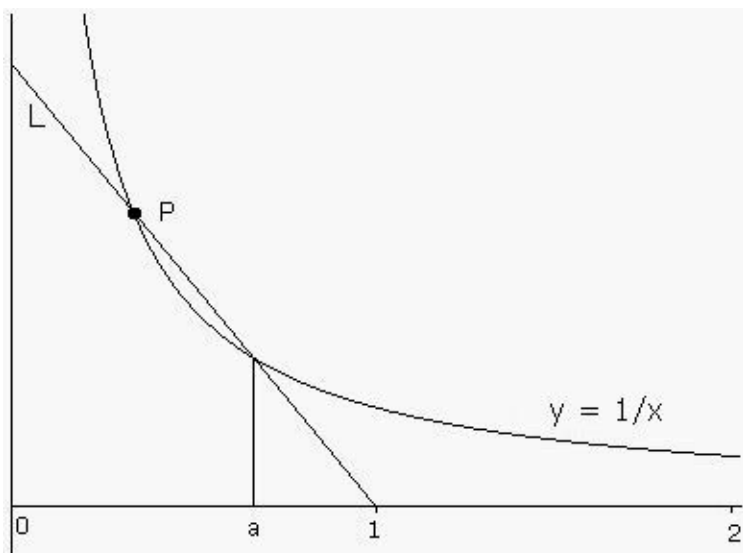


Note: For (b), give the area of the circle in terms of a , b and c .

2. A five gallon radiator contains a mixture of water and antifreeze.
- Suppose the mixture is currently 40% antifreeze. How much of the mixture must be drained, and replaced with *pure* antifreeze, so that the mixture will become a 50% antifreeze solution?
 - Generalize part (a), as follows: suppose the mixture is currently $x\%$ antifreeze, and we desire a $y\%$ antifreeze solution. In terms of x and y , how much of the mixture must be drained, and replaced with pure antifreeze, to achieve a $y\%$ solution?

3. Two motorists set out at the same time from A to B, a distance of 100 miles. They follow the same route but travel at different speeds. Each travels at a constant rate which is an integer number of miles per hour, and their speeds differ by a prime number of miles per hour. After driving for two hours, the distance of the slower car from A is five times that of the faster car from B. What is the speed of each motorist?
4. A box contains 5 computer memory chips, 2 of which are defective. The chips are tested, one at a time, until the second defective chip is found.
- (a) What is the probability that the fifth chip to be tested will be the second defective chip?
- (b) Can you generalize this probability for n chips? (That is, answer question (a), but under the assumption that the box contains n chips, with $n \geq 5$, of which 2 are defective.)

5. Refer to the diagram below. Assume that the curve $y = \frac{1}{x}$ intersects the line L at $x = a$, and that L intersects the x -axis at $x = 1$. Assume also that L and $y = \frac{1}{x}$ both pass through point P .

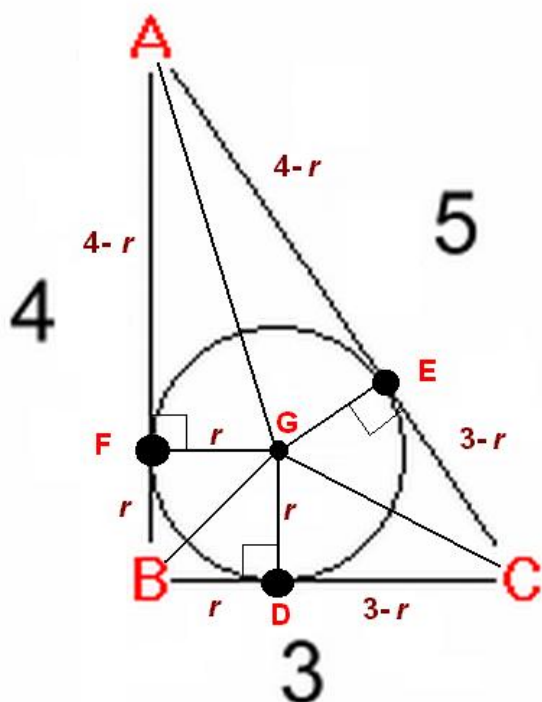


- (a) Find the y -intercept of line L .
- (b) Find the slope of line L .
- (c) Identify the coordinates of point P .

Partial solutions:

1. **Solution** to part (a): π

Proof: The below diagram is an enlarged version of the original, with a few added points and line segments. G is the center of the inscribed circle, and points D , E , F are the points where the sides of the triangle are tangent to the circle. Therefore, segments GD , GE and GF are radii of the inscribed circle, and so these segments are perpendicular to the sides of a triangle. (Recall from geometry that a radius and a tangent are always perpendicular at the point of tangency.)



Since GD , GE and GF are all radii of the same circle, they have the same length, which we call r . It follows that quadrilateral $FDGB$ is a square, and so segment BF also has length r . It now follows easily that segment DC has length $3 - r$ (since segment AC 's length is 3), and segment FA has length $4 - r$.

Next, observe that segments AE and AF must have the same length. (Recall, again from Euclidean geometry, that when two tangents to a circle are drawn from a common exterior point – A , in this case – the distances from that point to the two points of tangency are equal.) Similarly, segments CE and CD must have the same length as well. Therefore, segments AE and EC have lengths $4 - r$ and $3 - r$, respectively.

Finally, notice that segments AE and EC combine to form segment AC , whose length is 5. Therefore, we must have $(4 - r) + (3 - r) = 5$, which gives us $r = 1$, and thus the area is π .

(Note: for part (b), the radius of the inscribed circle turns out to be $\frac{b + a - c}{2}$, and therefore its area is $\pi \left(\frac{b + a - c}{2} \right)^2$. This may be shown by using the above method, with a, b, c instead of 3,4,5.)

2. As with #1, we will provide a solution for the specific case (part (a)), which may be adapted to solve the general case (part (b)).

We are given that the 5 gallon solution includes 2 gallons of antifreeze (since 40% of 5 is 2). Let x denote the amount of solution that must be drained and replaced with pure antifreeze in order to create a 50% solution (meaning 2.5 gallons of antifreeze). Since 40% of the drainage will be antifreeze, we're removing $0.4x$ gallons of antifreeze by draining x gallons of solution. Then, we are adding x gallons of pure antifreeze. We must choose x such that $2 - 0.4x + x = 2.5$. The solution to this equation is $x = \frac{5}{6}$; therefore, we must drain $\frac{5}{6}$ of a gallon of the mixture, and replace it with $\frac{5}{6}$ of pure antifreeze, to achieve a 50% solution.

3. Let s and f denote the speed of the slower car and the speed of the faster car, respectively, measured in miles per hour. After two hours, then, the cars have travelled $2s$ and $2f$ miles, respectively. Keep in mind that B is 100 miles from A, though, so the distance of the faster car *from B* after 2 hours is $100 - 2f$.

The sentence "After driving for two hours, the distance of the slower car from A is five times that of the faster car from B" translates into the language of algebra as $2s = 5(100 - 2f)$, which may be written more simply as $s = 250 - 5f$.

Now we have expressed s as a function of f ; this gives us as many pairs (s,f) as we'd like – for example, if $f = 1$, then $s = 245$ – which would satisfy the equation. However, we must find such a pair (f,s) that satisfies the conditions laid out in the instructions: both s and f must be positive integers, s must be less than f , and $f - s$ must be prime.

It's not hard to determine (trial-and-error works here) that the only positive integer values of f that return a smaller positive value for s are $42 \leq f \leq 49$, which gives us the following possibilities for (f,s) : $(42,40)$, $(43,35)$, $(44,30)$, $(45,25)$, $(46,20)$, $(47,15)$, $(48,10)$, and $(49,5)$. The only one of these pairs for which $f - s$ is prime is the first one: $(42,40)$. Therefore, the cars' speeds are 42 mph and 40 mph.

(Alternative approach: substitute $s + p$ for f , where p is some prime number, so that equation $s = 250 - 5f$ becomes $s = 250 - 5(s + p)$. One could then solve for s , getting $s = \frac{250 - 5p}{6}$. Since s must be an integer, it follows that the numerator must be even, which is only possible for the prime $p = 2$.)

4. We will provide a solution for part (a); this approach may then be adapted to solve part (b).

Since the chips are chosen at random, all orderings of the chips are equally likely. In particular, the combination of chips that will be defective are equally likely to occur – for example, the event that the first and second chips are defective is exactly as likely as the event that the third and fourth chips are defective (for example). The number of such combinations possible is ${}_5C_2$ (sometimes written $\binom{5}{2}$), which is equal to $\frac{5!}{2!3!} = 10$.

At this point, it's actually simpler to count the number of such combinations for which the fifth chip is *not* defective. We do so by counting the number of ways that 2 of the first *four* chips (thus excluding the fifth chip) can be defective. This is found by calculating ${}_4C_2 = \frac{4!}{2!2!} = 6$. Therefore, there are 6 combinations (out of 10) in which the last chip is *not* defective, leaving 4 combinations in which the last chip *is* defective. Since there 4 out of 10 equally likely

combinations in which the fifth chip is defective, the probability of this occurring is $\frac{4}{10}$, which of course reduces to $\frac{2}{5}$.

This method, using combinations, can be used to solve part (b); the solution for n chips is $\frac{{}_n C_2 - (n-1) C_2}{{}_n C_2}$, which simplifies (verify this for yourself!) to $\frac{2}{n}$.

Note: A student who is unfamiliar with the formula for ${}_n C_r$ can still solve part (a) by simply *listing* all possible combinations, as shown below. (In the following list, + denotes a functioning chip, and - denotes a defective chip.)

- - + + +
 - + - + +
 - + + - +
 - + + + -
 + - - + +
 + - + - +
 + - + + -
 + + - - +
 + + - + -
 + + + - -

By counting the number of combinations (10) and the number which have a defective chip in the 5th position (4), one may arrive at the correct answer. (However, this method only works for a fixed number of chips, and would not work well for the general case.)

5. First, find an equation for line L . We are given that the points $\left(a, \frac{1}{a}\right)$ and $(1,0)$ are on the line. This allows us to calculate the slope of L :

$$\begin{aligned} \text{slope} &= \frac{0 - \frac{1}{a}}{1 - a} \\ &= \frac{1}{a^2 - a}. \end{aligned}$$

Next, using the point-slope formula for the equation of a line (with the above slope and the point $(1,0)$) gives us the equation for L :

$$\begin{aligned} y &= \frac{1}{a^2 - a} (x - 1) \\ &= \frac{1}{a^2 - a} x - \frac{1}{a^2 - a} \text{ (written in slope-intercept form)} \end{aligned}$$

This gives us the answers to parts (a) and (b).

For part (c), we must find the values of x for which curve $y = \frac{1}{x}$ and line L intersect. Set up the equation, and solve for x ; you should get two solutions: $x = a$, and $x = 1 - a$. Therefore, the coordinates of point P are $\left(1 - a, \frac{1}{1 - a}\right)$.