

The Twenty-Seventh Annual

Eastern Shore High School Mathematics Competition

November 10, 2010

Team Contest Exam

Instructions

Answer as many questions as possible in the time provided. To receive full credit for a correct solution, show all work and provide a clearly written explanation. Solutions will be judged based on correctness, completeness and clarity. (Little credit, if any, will be given for a solution consisting of just a number or a single sentence.)

All work and answers must be written on the provided sheets of plain white paper. Use only one side of each sheet of paper, and start each new problem on a new sheet of paper.

1. Your team has been provided with a set of snap cubes. For the following instructions, consider each snap cube to be one cubic unit - that is, each snap cube measures 1 unit by 1 unit by 1 unit (or just " $1 \times 1 \times 1$," for short). For example, a cube that is constructed using eight snap cubes would measure 2 units by 2 units by 2 units (or " $2 \times 2 \times 2$ ").
 - (a) Use eight snap cubes to construct a $2 \times 2 \times 2$ cube. If we were to *paint* the exterior of this $2 \times 2 \times 2$ cube, what would be the correct answer to each of the following questions?
 - i. How many of the snap cubes have paint on exactly one side?
 - ii. How many of the snap cubes have paint on exactly two sides?
 - iii. How many of the snap cubes have paint on exactly three sides?
 - iv. How many of the snap cubes have paint on more than three sides?
 - (b) Use 27 snap cubes to construct a $3 \times 3 \times 3$ cube. If we were to paint the exterior of this $3 \times 3 \times 3$ cube, what would be the correct answer to each of the following questions?
 - i. How many of the snap cubes have paint on exactly one side?
 - ii. How many of the snap cubes have paint on exactly two sides?
 - iii. How many of the snap cubes have paint on exactly three sides?
 - iv. How many of the snap cubes have paint on more than three sides?
 - (c) Use 64 snap cubes to construct a $4 \times 4 \times 4$ cube. If we were to paint the exterior of this $4 \times 4 \times 4$ cube, what would be the correct answer to each of the following questions?
 - i. How many of the snap cubes have paint on exactly one side?
 - ii. How many of the snap cubes have paint on exactly two sides?
 - iii. How many of the snap cubes have paint on exactly three sides?
 - iv. How many of the snap cubes have paint on more than three sides?
 - (d) Now that you've got the idea - imagine using n^3 snap cubes to construct an $n \times n \times n$ cube (where n may be any counting number). If we were to paint the exterior of this $n \times n \times n$ cube, what would be the correct answer (in terms of n) to each of the following questions?
 - i. How many of the snap cubes have paint on exactly one side?
 - ii. How many of the snap cubes have paint on exactly two sides?
 - iii. How many of the snap cubes have paint on exactly three sides?
 - iv. How many of the snap cubes have paint on more than three sides?

Note: For parts (a), (b), and (c), you may just write your answers based on your work with the snap cubes. However, for part (d), you will need to explain the reasoning behind your answers.

2. Solve for x in terms of a :

$$2 \left(\frac{x-1}{x-a} \right)^2 - 5 \left(\frac{x-1}{x-a} \right) + 2 = 0.$$

3. Mohammad's house is on a road where the house numbers run 1, 2, 3, 4, ..., consecutively. By a curious coincidence, the sum of all house numbers less than his house number is the same as the sum of all house numbers greater than his house number. Assuming Mohammad's house number is in the thirties, what is his house number, and how many houses are on his road?

4. (Loosely based on a true story.) On The Parkway, the toll at any exit can be paid with a token, regardless of the current toll charge or of how long ago the token was purchased.

Prior to 1989, the toll at each Parkway exit had been 25 cents; accordingly, the Parkway Authority charged \$10 for a roll of 40 tokens. On January 1, 1989, the Parkway Authority increased exit tolls from 25 cents to 35 cents, but continued to sell rolls of 40 tokens for \$10 per roll. At this time, they announced that on April 1, 1989, they would reduce the number of tokens per roll from 40 to 30 (while still charging \$10 per roll). Shortly after this announcement, there was a sudden increase in demand for tokens; as a result, the toll-takers, who sold the tokens, were frequently out of them.

Imagine that it is January 2, 1989, you are a person who drives on the Parkway regularly, and you have just learned about the planned price increase on tokens. Since tokens are cheaper now than they will be on April 1, you need to decide how many rolls of tokens to buy now at the lower price.

Assume that any money you do not spend on rolls of tokens could instead be invested in a savings account that will pay 4% interest compounded annually. Also, assume that each token you buy can be used forever – that is, tokens do not expire. Based on these assumptions, how many rolls of tokens should you buy?

(Hint: your answer may depend in part on information not given here. To account for this, you may want make a reasonable assumption, or - preferably - use a variable to represent an unknown quantity. Include this assumption and/or variable in your written work, and make it clear how this factors into your answer.)