1. A student's grade in an introductory statistics course is determined by grades on quizzes, homework, tests, and a final exam. The professor counts the average of 3 tests as 50%, quizzes as 20 %, homework as 10%, and the final exam as 20% of the final grade. Johnny has a test average of 78, a quiz average of 89, and homework average of 95, and he gets an 82 on the final exam. What is Johnny's final grade in the course?

   (a) 82.7 (b) 83.33 (c) 85 (d) 86 (e) Not enough information is provided

   Answer: (a) 82.7. This is because $0.5(78) + 0.2(89) + 0.1(95) + 0.2(82) = 82.7$.

2. Determine the volume of a rectangular prism if the areas of three of its faces are 6 cm^2, 8 cm^2 and 12 cm^2.

   (a) 24 cm^3 (b) 52 cm^3 (c) 72 cm^3 (d) 576 cm^3 (e) None of these

   Answer: (a) 24 cm^3. The given face areas imply that the dimensions of the prism are 2cm by 3cm by 4cm, since $6 = 2 \times 3$, $8 = 2 \times 4$, and $12 = 3 \times 4$. Thus, the volume of the prism is $2 \times 3 \times 4 = 24$ cm^3.

3. If $f(x) = \sqrt{x+4}$, then which of the following is equal to $f(x^2)$?

   (a) $x+2$ (b) $\pm(x+2)$ (c) $x+4$ (d) $x+16$ (e) None of these

   Answer: (e) None of these. $f(x^2) = \sqrt{x^2+4}$, which is not equivalent to any of the choices.

4. Which of the following is equal to $\dfrac{(3x^2yz^3)^2}{(x\sqrt{y})^3}$?

   (a) $x^3z^5\sqrt{y}$ (b) $3xz^6\sqrt{y^3}$ (c) $6xz^5\sqrt{y}$ (d) $6xz^6\sqrt{y}$ (e) $9xz^6\sqrt{y}$

   Answer: (e) $9xz^6\sqrt{y}$

$$
\begin{aligned}
\frac{(3x^2yz^3)^2}{(x\sqrt{y})^3} &= \frac{9x^4y^2z^6}{xy^{3/2}} \\
&= 9x^{(4-1)}y^{(2-1/2)}z^6 \\
&= 9x^3y^{1/2}z^6 \\
&= 9x^3z^6\sqrt{y}
\end{aligned}
$$

   Comment: At a glance, one can see that the numerical coefficient of the given expression will be $\dfrac{3^2}{1} = 9$. Thus, the only possible correct answer among the five choices given is (e), since it's the only one with a coefficient of 9.

5. If the length of the diagonal of a square is doubled, what is the ratio of the square's new area to its old area?

   (a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2 (e) 4

   Answer: (e) 4. If a square's length is doubled, then the length of each side is doubled as well. If the original side length was x, then the original square's area was x^2, while the new square's area is $(2x)^2$. This is $4x^2$, which is 4 times the original area.

6. Evaluate the expression $\dfrac{6x-30}{3x^3-24} \div \left(\dfrac{10x-29}{x^3-8} - \dfrac{x+2}{x^2+2x+4}\right)$ when $x = \sqrt{5} \cdot 5^{-1/2}$.

(a) 1/5 (b) 1/4 (c) 1/3 (d) 1/2 (e) 1

Answer: (d) 1/2. Note that $x = \sqrt{5} \cdot 5^{-1/2} = \dfrac{\sqrt{5}}{\sqrt{5}} = 1$. So,

$$\begin{aligned}
\frac{6x-30}{3x^3-24} \div \left(\frac{10x-29}{x^3-8} - \frac{x+2}{x^2+2x+4} \right) &= \frac{6-30}{3-24} \div \left(\frac{10-29}{1-8} - \frac{3}{x+2+4} \right) \\
&= \frac{-24}{-21} \div \left(\frac{-19}{-7} - \frac{3}{7} \right) \\
&= \frac{8}{7} \div \frac{16}{7} \\
&= \frac{8}{7} \times \frac{7}{16} \\
&= \frac{8}{16} = \frac{1}{2}.
\end{aligned}$$

7. It is known that for any data set, the proportion of observations that lie within 2 standard deviations around the mean is at least 3/4, or 75%. It is also known that at least 8/9 (89%) of observations lie within 3 standard deviations around the mean.

   Suppose the coach of a basketball team reports that the mean height of players on his team is 71.5 inches and the standard deviation is 1.7 inches.

   Consider the statements below:

   1. At least 75% of the players on the team are between 68.1 and 74.9 inches tall.
   2. At least 89% of the players on the team are between 69 and 73 inches tall.
   3. At most 25% of the players on the team are below 68.1 inches tall.
   4. At most 11% of the players on the team are above 76.6 inches tall.

   Which of these four statements is/are true?

   (a) Only 1 (b) Only 2 (c) Only 1 & 2 (d) Only 3 & 4 (e) 1, 3 & 4 but not 2

   Comment: The instructions for this problem outline what is known in statistics as Chebyshev's Rule.

   Answer: (e) 1, 3, & 4 but not 2. Note that the standard deviations fall as follows:

   - Two standard deviations below the mean: $71.5 - 2 \times 1.7 = 68.1$
   - One standard deviation below the mean: $71.5 - 1.7 = 69.8$
   - One standard deviation above the mean: $71.5 + 1.7 = 73.2$
   - Two standard deviations above the mean: $71.5 + 2 \times 1.7 = 74.9$
   - Three standard deviations above the mean: $71.5 + 3 \times 1.7 = 76.6$

   Based on these "cutoffs" for standard deviations, the description of Chebyshev's Rule indicates that statements 1, 3 and 4 are true, but not 2. For statement number 2, if we replaced "69 and 73" with "68.1 and 74.9" (or any interval containing this one), then the statement would have been true.

8.

5	78
6	44578
7	34466889999
8	0357789
9	01368

Above is a stem-and-leaf display of test scores. What score is the mode?

(a) 4 (b) 9 (c) 70 (d) 79 (e) 98

Answer: (d) 79. The four 9's in row 7 indicate that the number 79 appears four times in the data set. This is the most repetitions of any one value, making it the mode of the data set.

9. Which of the following is equal to $4^{\log_2 8} - 8^{\log_2 4}$?

(a) -4 (b) 0 (c) $2^{\log_4 8}$ (d) $2^{\log_8 4}$ (e) 4

Answer: (b) 0

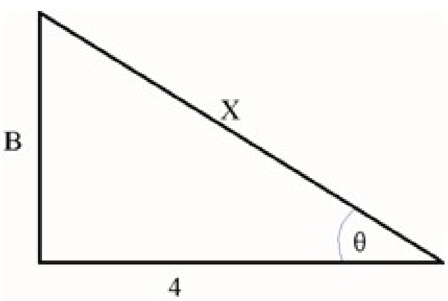
First, note that $\log_2 8 = \log_2(2^3) = 3$, and $\log_2 4 = \log_2(2^2) = 2$. Therefore,

$$4^{\log_2 8} - 8^{\log_2 4} = 4^3 - 8^2 = 64 - 64 = 0.$$

10. If θ is in the first quadrant and $\sec\theta = x/4$, then which of the following is equal to $\sin\theta$?

(a) $\dfrac{\sqrt{16-x^2}}{x}$ (b) $\dfrac{x-4}{4}$ (c) $\dfrac{\sqrt{x^2-16}}{x}$ (d) $\sqrt{16-x^2}$ (e) $\sqrt{x^2-16}$

Answer: (c) $\dfrac{\sqrt{x^2-16}}{x}$



Since θ is in the first quadrant you can construct a triangle with angle θ where $\sec\theta$ is the ratio of the lengths of the hypotenuse and adjacent sides. That is: $\sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{x}{4}$. It follows that $\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{B}{x}$. By the Pythagorean Theorem, $B = \sqrt{x^2-16}$, so $\sin\theta = \dfrac{\sqrt{x^2-16}}{x}$.

11. An equilateral triangle is inscribed in a circle of radius r. A smaller circle is inscribed in the inscribed triangle. This process is continued forever with progressively smaller circles. What is the sum of the perimeters of all of the triangles?

(a) $\dfrac{3r\sqrt{3}}{2}$ (b) $\dfrac{9r}{2}$ (c) $\pi r^2\sqrt{3}$ (d) $\dfrac{9r\sqrt{3}}{2}$ (e) $6r\sqrt{3}$

Answer: (e) $6r\sqrt{3}$

The perimeter of the first inscribed triangle is $3r\sqrt{3}$, and the circumference of the first inscribed circle is $r/2$. Since the inscribed circle's radius is half that of the original circle's radius, its inscribed triangle's perimeter will be half of the original triangle's perimeter. (This follows from the principle of geometric similarity.) This pattern will repeat - as a result, each triangle's perimeter will be one half of the previous triangle's perimeter. Therefore, the sum of the perimeters of the triangles will be:

$$\begin{aligned}
Sum &= 3\sqrt{3} + \frac{1}{2} \cdot 3r\sqrt{3} + \frac{1}{2} \cdot \frac{1}{2} \cdot 3r\sqrt{3} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 3r\sqrt{3} + \ldots \\
&= \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots\right) 3r\sqrt{3} \\
&= \left(\frac{1}{1-\frac{1}{2}}\right) 3r\sqrt{3} \\
&= 2 \cdot 3r\sqrt{3} \\
&= 6r\sqrt{3}
\end{aligned}$$

Note that we used the formula $1 + r + r^2 + r^3 + \ldots = \dfrac{1}{1-r}$ (which is valid when $|r| < 1$) to determine that the infinite series $\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots\right)$ converges to exactly 2.

12. A fair coin will be tossed four times. What is the probability that at least one of the four coin tosses will result in a head?

    (a) 1/4 (b) 3/8 (c) 1/2 (d) 3/4 (e) 15/16

    Answer: (e) 15/16

    The only way *not* to observe the event "at least one of the four coin tosses will result in a head" is to observe the event "all four coin tosses will result in tails." The latter event's probability is much easier to find directly than the former event, since it describes the joint occurrence of four independent events:

    $$P(4\text{ tails}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}.$$

    Since the probability of observing zero heads (that is, four tails) is 1/16, the probability of observing at least one head is 1-1/16, or 15/16.

13. Which of the following is equal to $e^{\ln x - \ln(x-1)} - e^{\ln x - \ln(x+1)}$?

    (a) $e^{\ln(x+1)-\ln(x-1)}$ (b) 2 (c) $\dfrac{x+1}{x-1}$ (d) $\dfrac{2x}{x^2-1}$ (e) e^2

    Answer: (d) $\dfrac{2x}{x^2-1}$

$$
\begin{aligned}
e^{\ln x-\ln(x-1)} - e^{\ln x-\ln(x+1)} &= \frac{e^{\ln x}}{e^{\ln(x-1)}} - \frac{e^{\ln x}}{e^{\ln(x+1)}} \\
&= \frac{x}{x-1} - \frac{x}{x+1} \\
&= \frac{x(x+1)-x(x-1)}{(x-1)(x+1)} \\
&= \frac{x^2+x-x^2+x}{x^2-1} \\
&= \frac{2x}{x^2-1}
\end{aligned}
$$

14. What digit is in the units place of the decimal representation of 3^{2010}?

    (a) 0 (b) 1 (c) 3 (d) 7 (e) 9

    Answer: (e) 9

    The key to solving this problem is to observe that the units digits of 3^4 is 1, since $3^4 = 81$. Since the units digit of 3^4 is 1, it follows that the units digit of any power of 3^4 will also be 1, since the product of any two numbers whose unit digit is 1 will always have a units digit of 1. In particular, $\left(3^4\right)^{502} = 3^{2008}$ has a units digit of 1, and so $3^{2010} = 3^2 \cdot 3^{2008} = 9 \cdot 3^{2008}$ has a units digit of $9 \cdot 1 = 9$.

    Another way of thinking about this problem is as follows: a bit of experimentation shows that the units digits of powers of 3, in sequential order, form the repeating pattern: 3, 9, 7, 1, 3, 9, 7, 1, ... Since every fourth entry in this list will be 1, it follows that the 2008^{th} entry will be 1, since 2008 is a multiple of 4. Therefore, starting from the 2008^{th} entry, it follows that 3^{2009} has a units digit of 3, and 3^{2010} has a units digit of 9.

15. Which of the following statements is/are logically equivalent to "Every widgit wobbles?"

    I. If it is a widgit, it wobbles.
    II. If it wobbles, it is a widgit.
    III. If it doesn't wobble, it is not a widgit.
    IV. If it is not a widgit, it doesn't wobble.

    (a) I only (b) II only (c) I and III (d) II and IV (e) All of them

    Answer: (c) I and III

    The statement "Every widgit wobbles" is directly equivalent to "If it is a widgit, then it wobbles." Every conditional ("if/then") statement is logically equivalent to its contrapositive – that is, the statement "If A, then B" is true if and only if the statement "If not B, then not A" is true. The contrapositive of the given statement is "If it doesn't wobble, then it is not a widgit." Thus, statements I and III are logically equivalent to the original statement.

    Note: II is the "converse" ("If B, then A") of the original statement, and IV is the "inverse" ("If not A, then not B") of the original statement. A conditional statement is not logically equivalent to its converse or to its inverse. This is because it's possible for a conditional statement to be true while its converse (or inverse) is false, or vice-versa. For example, if we came across something that is not a widgit but still wobbles, then the original statement would still be considered true in that case, but the converse (statement II) would be false.

16. How many solutions are there to the equation $\cos\phi = \frac{1}{2}$ on the interval $-2\pi \le \phi \le 2\pi$?

(a) 1 (b) 2 (c) 4 (d) 8 (e) infinitely many

Answer: (c) 4

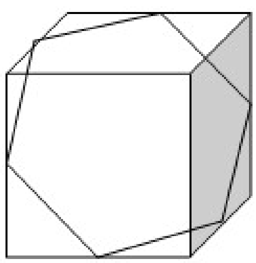
The "primary" solution is $\phi = \pi/3$, since $\pi/3$ (radians) is equivalent to $60°$, and it is well known that $\cos 60° = 1/2$. Since cos is an even function, it follows that $\cos(-\pi/3)$ is also $1/2$. Since cos is periodic with period 2π, it follows that $\pi/3 - 2\pi = -5\pi/3$ and $-\pi/3 + 2\pi = 5\pi/3$ are also solutions. Thus, there are four solutions to the equation $\cos\phi = \frac{1}{2}$ on the interval $-2\pi \le \phi \le 2\pi$.

Note: if one draws a reasonably accurate graph of the cosine function on the interval $-2\pi \le \phi \le 2\pi$, then it is immediately clear that the graph intersects the horizontal line $y = 1/2$ in four places on that interval.

17. A unit cube can be cut by a plane such that the cross section formed is a regular hexagon. Determine the perimeter of the regular hexagon.

(a) $2\sqrt{2}$ (b) 3 (c) $2\sqrt{3}$ (d) $3\sqrt{2}$ (e) 6

Answer: (d) $3\sqrt{2}$



The plane passes through the midpoints of the indicated sides. For each "corner right triangle:" $(1/2)^2 + (1/2)^2 = (\text{length of hypotenuse})^2$. Therefore, the length of each hypotenuse is $\dfrac{\sqrt{2}}{2}$, which means the regular hexagon's perimeter is $6 \times \dfrac{\sqrt{2}}{2} = 3\sqrt{2}$.

18. (Tiebreaker #1) Consider the equation

$$\sqrt{\frac{20+x}{x}} + \sqrt{\frac{20-x}{x}} = \sqrt{6}.$$

One solution of this equation is $x = 12$. Which is of the following statements is also true about this equation? (Note: in choices (c) and (d) below, i denotes the "imaginary unit," for which $i^2 = -1$.)

a) Its other solution is a negative real number.
b) Its other solution is a non-negative real number.
c) Its other solution is of the form $a + bi$, where a and b are real numbers and $b > 0$.
d) Its other solutions are of the form $a \pm bi$, where a and b are non-zero real numbers.
e) It has no solution other than $x = 12$.

Answer: (e) It has no solution other than $x = 12$.

To solve this equation for x, it helps (a lot) to first make the substitution $a = \frac{20}{x}$. With this substitution, $\frac{20+x}{x}$ and $\frac{20-x}{x}$ can be written as $a+1$ and $a-1$, respectively. Thus, the algebra proceeds as follows:

$$\begin{aligned}
\sqrt{a+1} + \sqrt{a-1} &= \sqrt{6} \\
\left(\sqrt{a+1} + \sqrt{a-1}\right)^2 &= 6 \\
(a+1) + (a-1) + 2\sqrt{(a+1)(a-1)} &= 6 \\
2a + 2\sqrt{a^2-1} &= 6 \\
2\sqrt{a^2-1} &= 6 - 2a \\
\sqrt{a^2-1} &= 3 - a \\
a^2 - 1 &= (3-a)^2 \\
a^2 - 1 &= 9 - 6a + a^2 \\
6a &= 10 \\
a &= \frac{5}{3}
\end{aligned}$$

Thus, we have $a = 5/3$, which means $20/x = 5/3$. The solution to this equation is $x = 12$. Since there are no other solutions to the equation, choice (e) is the best answer for this problem.

19. (Tiebreaker #2) Let F be a function from set A to set B.
If C is a subset of A, then we write "$F(C)$" to denote the set $\{F(x)|x \in C\}$.
(The set $F(C)$ is called the "image of C under F.") Note that $F(C)$ is a subset of B.

Select the choice which makes the statement labeled (*) below a true statement for all functions F from A to B. (Or, select choice (e) if none of the other choices is valid.)

(*) $F(S \cap R) = F(S) \cap F(R)$ for all subsets S, R of A...

a) if and only if F is one-to-one.
b) if and only if F is onto.
c) if and only if F is both one-to-one and onto.
d) only if S and R are disjoint.
e) None of these

Answer: (a) if and only if F is one-to-one.

First note that if $x \in S \cap R$, then $x \in S$, which implies $F(x) \in F(S)$. Similarly, if $x \in S \cap R$, then $x \in R$, which implies $F(x) \in F(R)$. Thus, $F(S \cap R)$ is contained in both $F(S)$ and $F(R)$, so $F(S \cap R) \subseteq F(S) \cap F(R)$. This is true for *any* function F. So, the question is what additional restrictions (if any) are needed to ensure that $F(S) \cap F(R) \subseteq F(S \cap R)$.

Suppose $y \in B$ such that $y \in F(S) \cap F(R)$; that is, y is in both $F(S)$ and $F(R)$. That means y is in the image of S, so there is some $x_1 \in S$ such that $F(x_1) = y$. Also, y is in the image of R, so there is some $x_2 \in R$ such that $F(x_2) = y$. Thus, $F(x_1) = F(x_2)$.

Now, if F is one-to-one, then $F(x_1) = F(x_2)$ implies $x_1 = x_2$; this implies $x_1 \in R \cap S$, which means $y \in F(R \cap S)$. So, if F is one-to-one, then $y \in F(S \cap R)$.

On the other hand, if F is *not* one-to-one, then it's possible that $x_1 \neq x_2$. This allows the possibility that $x_1 \in S$, $x_2 \in R$, but neither x_1 nor x_2 is in $S \cap R$. Therefore, if F is not one-to-one, then it is not necessarily true that $F(S \cap R) \subseteq F(S) \cap F(R)$.

20. (Tiebreaker #3) Today is November 10, 2010; if we write this in the standard "mm/dd/yy" (month/day/year) format, then today is "11/10/10." Define a day's "date product" as the product of the month number, day number, and two-digit year number (e.g. 10, rather than 2010) of that date; for example, today's "date product" would be $11 \times 10 \times 10 = 1100$. Similarly, tomorrow's "date product" will be $11 \times 11 \times 10 = 1210$, since tomorrow's date will be 11/11/10.

    During the course of one century (that is, between 01/01/01 and 12/31/00), how many days have a "date product" of 1210?

    (a) 9 (b) 10 (c) 11 (d) 12 (e) 13

    Answer: (b) 10

    To solve this problem, it helps to first list all of the possible values for mm, dd, and yy, each of which must be a factor of 1210:

    $mm \in \{1, 2, 5, 10, 11\}$, since $mm \leq 12$
    $dd \in \{1, 2, 5, 10, 11, 22\}$, since $dd \leq 31$
    $yy \in \{1, 2, 5, 10, 11, 22, 55\}$, since $yy \leq 99$

    We need to find triplets (mm, dd, yy), subject to the above restrictions, such that $mm \times dd \times yy = 1210$. One strategy is to look for eligible dates mm/dd in each available year:

    - yy=01: We would need to find mm and dd such that $mm \times dd = 1210$. Since the maximum values for mm and dd are 11 and 22, respectively, the maximum possible product of mm and dd is 242. Therefore, no date in year yy=01 has a date product of 1210.
    - yy=02: Similar problem - we'd need to have $mm \times dd = 605$, which is impossible.
    - yy = 05: The one date in year yy=05 with date product 1210 is the date on which $mm \times dd = 242$: 11/22/05 is our first solution.
    - yy=10: We need to find values of mm and dd such that $mm \times dd = 121$. The only such solution is 11/11/10 (tomorrow!).
    - yy=11: We need to find values of mm and dd such that $mm \times dd = 110$. This happens three times: 5/22/11, 10/11/11, and 11/10/11 (one year from today!).
    - yy=22: We need to find values of mm and dd such that $mm \times dd = 55$. This happens twice: 5/11/22 and 11/5/22.

- yy=55: We need to find values of mm and dd such that $mm \times dd = 22$. This happens three times: 1/22/55, 2/11/55, and 11/2/55.

Thus, there are ten dates per century with a date product of 1210: 11/22/05, 11/11/10, 5/22/11, 10/11/11, 11/10/11, 5/11/22, 11/5/22, 1/22/55, 2/11/55, and 11/2/55.