The Twenty-Seventh Annual

Eastern Shore High School Mathematics Competition

November 10, 2010

Individual Contest Exam

Instructions

Select the best answer for each question on this exam.

Your score will be the number of *correct* answers that you select.
There is no penalty for incorrect answers.

The use of a calculator is *not* permitted on this exam.

In the event of tie scores, questions #18, #19 and #20 will be used as tiebreakers.

1. A student's grade in an introductory statistics course is determined by grades on quizzes, homework, tests, and a final exam. The professor counts the average of 3 tests as 50%, quizzes as 20 %, homework as 10%, and the final exam as 20% of the final grade. Johnny has a test average of 78, a quiz average of 89, and homework average of 95, and he gets an 82 on the final exam. What is Johnny's final grade in the course?

   (a) 82.7 (b) 83.33 (c) 85 (d) 86 (e) Not enough information is provided

2. Determine the volume of a rectangular prism if the areas of three of its faces are 6 cm^2, 8 cm^2 and 12 cm^2.

   (a) 24 cm^3 (b) 52 cm^3 (c) 72 cm^3 (d) 576 cm^3 (e) None of these

3. If $f(x) = \sqrt{x+4}$, then which of the following is equal to $f(x^2)$?

   (a) $x+2$ (b) $\pm(x+2)$ (c) $x+4$ (d) $x+16$ (e) None of these

4. Which of the following is equal to $\dfrac{(3x^2yz^3)^2}{(x\sqrt{y})^3}$?

   (a) $x^3z^5\sqrt{y}$ (b) $3xz^6\sqrt{y^3}$ (c) $6xz^5\sqrt{y}$ (d) $6xz^6\sqrt{y}$ (e) $9xz^6\sqrt{y}$

5. If the length of the diagonal of a square is doubled, what is the ratio of the square's new area to its old area?

   (a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2 (e) 4

6. Evaluate the expression $\dfrac{6x-30}{3x^3-24} \div \left(\dfrac{10x-29}{x^3-8} - \dfrac{x+2}{x^2+2x+4}\right)$ when $x = \sqrt{5}\cdot 5^{-1/2}$.

   (a) 1/5 (b) 1/4 (c) 1/3 (d) 1/2 (e) 1

7. It is known that for any data set, the proportion of observations that lie within 2 standard deviations around the mean is at least 3/4, or 75%. It is also known that at least 8/9 (89%) of observations lie within 3 standard deviations around the mean.

   Suppose the coach of a basketball team reports that the mean height of players on his team is 71.5 inches and the standard deviation is 1.7 inches.

   Consider the statements below:

   1. At least 75% of the players on the team are between 68.1 and 74.9 inches tall.
   2. At least 89% of the players on the team are between 69 and 73 inches tall.
   3. At most 25% of the players on the team are below 68.1 inches tall.
   4. At most 11% of the players on the team are above 76.6 inches tall.

   Which of these four statements is/are true?

   (a) Only 1 (b) Only 2 (c) Only 1 & 2 (d) Only 3 & 4 (e) 1, 3 & 4 but not 2

8.

5	78
6	44578
7	34466889999
8	0357789
9	01368

Above is a stem-and-leaf display of test scores. What score is the mode?

(a) 4 (b) 9 (c) 70 (d) 79 (e) 98

9. Which of the following is equal to $4^{\log_2 8} - 8^{\log_2 4}$?

(a) -4 (b) 0 (c) $2^{\log_8 4}$ (d) $2^{\log_4 8}$ (e) 4

10. If θ is in the first quadrant and $\sec\theta = x/4$, then which of the following is equal to $\sin\theta$?

(a) $\dfrac{\sqrt{16-x^2}}{x}$ (b) $\dfrac{x-4}{4}$ (c) $\dfrac{\sqrt{x^2-16}}{x}$ (d) $\sqrt{16-x^2}$ (e) $\sqrt{x^2-16}$

11. An equilateral triangle is inscribed in a circle of radius r. A smaller circle is inscribed in the inscribed triangle. This process is continued forever with progressively smaller circles. What is the sum of the perimeters of all of the triangles?

(a) $\dfrac{3r\sqrt{3}}{2}$ (b) $\dfrac{9r}{2}$ (c) $\pi r^2\sqrt{3}$ (d) $\dfrac{9r\sqrt{3}}{2}$ (e) $6r\sqrt{3}$

12. A fair coin will be tossed four times. What is the probability that at least one of the four coin tosses will result in a head?

(a) 1/4 (b) 3/8 (c) 1/2 (d) 3/4 (e) 15/16

13. Which of the following is equal to $e^{\ln x - \ln(x-1)} - e^{\ln x - \ln(x+1)}$?

(a) $e^{\ln(x+1)-\ln(x-1)}$ (b) 2 (c) $\dfrac{x+1}{x-1}$ (d) $\dfrac{2x}{x^2-1}$ (e) e^2

14. What digit is in the units place of the decimal representation of 3^{2010}?

(a) 0 (b) 1 (c) 3 (d) 7 (e) 9

15. Which of the following statements is/are logically equivalent to "Every widgit wobbles?"

I. If it is a widgit, it wobbles.
II. If it wobbles, it is a widgit.
III. If it doesn't wobble, it is not a widgit.
IV. If it is not a widgit, it doesn't wobble.

(a) I only (b) II only (c) I and III (d) II and IV (e) All of them

16. How many solutions are there to the equation $\cos\phi = \frac{1}{2}$ on the interval $-2\pi \le \phi \le 2\pi$?

(a) 1 (b) 2 (c) 4 (d) 8 (e) infinitely many

17. A unit cube can be cut by a plane such that the cross section formed is a regular hexagon. Determine the perimeter of the regular hexagon.

(a) $2\sqrt{2}$ (b) 3 (c) $2\sqrt{3}$ (d) $3\sqrt{2}$ (e) 6

18. (Tiebreaker #1) Consider the equation

$$\sqrt{\frac{20+x}{x}} + \sqrt{\frac{20-x}{x}} = \sqrt{6}.$$

One solution of this equation is $x = 12$. Which is of the following statements is also true about this equation? (Note: in choices (c) and (d) below, i denotes the "imaginary unit," for which $i^2 = -1$.)

a) Its other solution is a negative real number.
b) Its other solution is a non-negative real number.
c) Its other solution is of the form $a + bi$, where a and b are real numbers and $b > 0$.
d) Its other solutions are of the form $a \pm bi$, where a and b are non-zero real numbers.
e) It has no solution other than $x = 12$.

19. (Tiebreaker #2) Let F be a function from set A to set B.
If C is a subset of A, then we write "$F(C)$" to denote the set $\{F(x)|x \in C\}$.
(The set $F(C)$ is called the "image of C under F.") Note that $F(C)$ is a subset of B.

Select the choice which makes the statement labeled (*) below a true statement for all functions F from A to B. (Or, select choice (e) if none of the other choices is valid.)

(*) $F(S \cap R) = F(S) \cap F(R)$ for all subsets S, R of A...

a) if and only if F is one-to-one.
b) if and only if F is onto.
c) if and only if F is both one-to-one and onto.
d) only if S and R are disjoint.
e) None of these

20. (Tiebreaker #3) Today is November 10, 2010; if we write this in the standard "mm/dd/yy" (month/day/year) format, then today is "11/10/10." Define a day's "date product" as the product of the month number, day number, and two-digit year number (e.g. 10, rather than 2010) of that date; for example, today's "date product" would be $11 \times 10 \times 10 = 1100$. Similarly, tomorrow's "date product" will be $11 \times 11 \times 10 = 1210$, since tomorrow's date will be 11/11/10.

During the course of the century – that is, from January 1, 2001 (01/01/01) through December 31, 2100 (12/31/00) – how many days are there with a "date product" of 1210?

(a) 9 (b) 10 (c) 11 (d) 12 (e) 13