

The Twenty-Sixth Annual

Eastern Shore High School Mathematics Competition

November 12, 2009

Individual Contest Exam

Instructions

Select the best answer for each question on this exam.

Your score will be the number of *correct* answers that you select.

There is no penalty for incorrect answers.

The use of a calculator is *not* permitted on this exam.

In the event of tie scores, questions #18, #19 and #20 will be used as tiebreakers.

1. The integer sequence defined by $d_1 = 1$, $d_2 = 3$, and $d_{n+1} = 2d_n + d_{n-1}$ for $n \geq 2$, is called the sequence of “diagonal numbers.” Which of the following is true?
- (a) Every diagonal number is a prime number.
 (b) Every diagonal number is an odd number.
 (c) Every diagonal number is the sum of two integer squares.
 (d) The seventh diagonal number is 99.
 (e) None of these.
2. Among 30 students taking a course in Calculus II, 15 students know the JAVA programming language, 12 students know Alice, 6 students know C++, 4 students know both C++ and Alice, 5 students know both JAVA and Alice, 3 students know both JAVA and C++, and 2 students know all three languages. How many students in the class know *only* JAVA?
- (a) 5 (b) 7 (c) 9 (d) 15 (e) 25
3. Over four consecutive years, the population of a town increased 10% during the first year, then increased 10% during the second year, then decreased 10% during the third year, then decreased 10% during the fourth year. To the nearest percent, what was the net change in the town’s population over this four year period?
- (a) -4% (b) -2% (c) 0% (d) 2% (e) 4%
4. What is the sum of the digits of the largest prime factor of 2009?
- (a) 5 (b) 8 (c) 11 (d) 13 (e) 17
5. Which of the following is equal to $512 - 256 + 128 - \dots + 2$?
- (a) 342 (b) 386 (c) 504 (d) 511 (e) 1023
6. Suppose A is a positive number. If $\log_2 A = k$ and $\log_8 A^3 = 9$, then what is the value of k ?
- (a) 3 (b) 4 (c) 9 (d) 8^9 (e) None of these
7. The expression
- $$\frac{1}{1 + \tan^2 \alpha} + \frac{1}{1 + \cot^2 \alpha}$$
- simplifies to which of the following?
- (a) 1 (b) $\sin \alpha$ (c) $\cos \alpha$ (d) $\frac{1}{\sin \alpha + \cos \alpha}$ (e) $\tan \alpha + \cot \alpha$
8. For the function $r(x) = x + \frac{1}{x}$, what is the value of the following expression?
- $$r(1) + r(2) - r\left(\frac{1}{2}\right) + r(4) - r\left(\frac{1}{4}\right) + r(6) - r\left(\frac{1}{6}\right) + r(8) - r\left(\frac{1}{8}\right) + r(10) - r\left(\frac{1}{10}\right)$$
- (a) 2 (b) 10 (c) 26 (d) 30 (e) 32

9. What value of x satisfies the following equation?

$$\sqrt{2^{\log_2 2^{\log_2 2^{\log_2 2^{\log_2 x}}}}} = 5$$

- (a) 5 (b) 10 (c) 16 (d) 25 (e) 32

10. A diagonal of a rectangular solid is defined to be a segment whose end points are two vertices not in the same face. If the diagonals of a cube are $4\sqrt{3}$ cm long, what is the volume of the cube (in cubic centimeters)?

- (a) $9\sqrt{3}$ (b) 27 (c) 64 (d) $48\sqrt{6}$ (e) $192\sqrt{3}$

11. A coin is weighted in such a way that it will come up heads 60% of the time. If this coin is flipped three times, what is the probability that it will come up heads fewer than two times?

- (a) 0.096 (b) 0.160 (c) 0.192 (d) 0.216 (e) 0.352

12. The Zyblonski number rule is this: starting with the fourth number, each Zyblonski number is the sum of the previous three. So, if the first three Zyblonski numbers were 1, 1, 1, then the following ten Zyblonski numbers would be 3, 5, 9, 17, 31, 57, 105, 193, 355, 653. Thus, we would say the thirteenth Zyblonski number is 653.

What would the thirteenth Zyblonski number be if the first three Zyblonski numbers were 1, 2, 3 rather than 1, 1, 1?

- (a) 377 (b) 655 (c) 1278 (d) 1431 (e) 1512

13. Which of the following is equal to $(\sin x + \cos x)^2$?

- (a) $1 + \sin x \cos x$ (b) $\sin^2 x + \cos^2 x$ (c) $1 + \sin 2x$ (d) $2(1 + \sin x \cos x)$
(e) None of these

14. Which of the following is true about the circles $x^2 + y^2 + 2y = 0$ and $(x - 1)^2 + (y + 2)^2 = 5$?

- a) They do not intersect at any point.
b) They are tangent to one another and one circle lies within the interior of the other circle.
c) They are tangent to one another and neither circle lies within the interior of the other circle.
d) They intersect at more than one point.

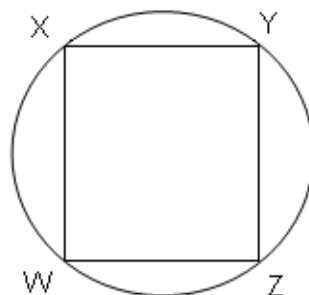
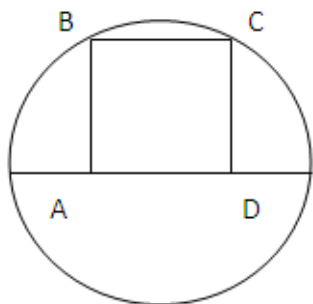
15. What is the value of $10^{3 \log 2 - 2 \log (\log 10^3) + \log ((\log 10^6)^2)}$?

- (a) 2 (b) 3 (c) 6 (d) 25 (e) 32

16. Using the fact that $75=30+45$, find the value of $\sin 75^\circ$.

- (a) $\frac{\sqrt{2}}{4}$ (b) $\frac{\sqrt{6}}{4}$ (c) $\frac{\sqrt{6} + \sqrt{2}}{4}$ (d) $\frac{1 + \sqrt{2}}{2}$ (e) $\frac{\sqrt{3} + \sqrt{2}}{2}$

17. In the diagram below, each of the circles has a 10 cm radius. Figure ABCD is a square inscribed in a semicircle. Figure WXYZ is a square inscribed in a circle. What is the ratio of the area of square ABCD to the area of square WXYZ?



- (a) $1/5$ (b) $2/5$ (c) $1/2$ (d) $3/5$ (e) $4/5$

18. (Tiebreaker #1) Each inhabitant of Tiebreak Island is either a truth-teller or a liar. You come across three inhabitants of the island. Rick says “Randy is telling the truth,” Randy says “Rick and Rose are lying,” and Rose says “Rick is lying.” What can you conclude?

- (a) All three are truth-tellers.
 (b) All three are liars.
 (c) Rick & Rose are truth-tellers; Randy is a liar.
 (d) Rick & Randy are truth-tellers; Rose is a liar.
 (e) Rose is a truth-teller; Rick and Randy are liars.

19. (Tiebreaker# 2) The equation $4^{2x-1} + 8 = 33 \cdot 4^{x-1}$ has one positive solution.

Which of the following intervals of real numbers contains this solution?

- (a) $[1,2)$ (b) $[2,3)$ (c) $[3,4)$ (d) $[4,5)$ (e) None of these

20. (Tiebreaker #3) K is a point lying outside of square MATH, so that $KM = 5$ cm, $KA = 4$ cm and $KT = 7$ cm. If points M, A, T, H and K are coplanar, find the length of segment TH.

- (a) 3 cm (b) $\sqrt{24}$ cm (c) $\sqrt{33}$ cm (d) 6 cm (e) 16 cm